Mathemat

THE MATHEMATICAL REPOSITORY.

BY

T. LEYBOURN.

VOL. I.

SECOND EDITION.

LONDON:

PRINTED AND SOLD BY W. GLENDINNING, NO. 9, CHARLES-STREET, HATTON-GARDEN.

1799.

TO

CHARLES HUTTON, Esq. L.L.D. F.R.S.

AND

PROFESSOR OF MATHEMATICS

IN THE

ROYAL MILITARY ACADEMY, WOOLWICH,

THIS VOLUME

OF THE

MATHEMATICAL REPOSITORY

IS

RESPECTFULLY DEDICATED,

AS A SMALL

TESTIMONY OF ESTEEM

FOR HIS

WORTH AND ABILITIES,

BY

THE EDITOR.

THE

Mathematical Repoſitory:

CONTAINING

MANY INGENIOUS AND USEFUL

ESSAYS AND EXTRACTS,

WITH A COLLECTION OF

PROBLEMS AND SOLUTIONS,

SELECTED FROM THE CORRESPONDENCE OF

SEVERAL ABLE MATHEMATICIANS,

AND THE WORKS OF THOSE WHO ARE EMINENT IN THE MATHEMATICS.

LONDON:

PRINTED FOR THE EDITOR;

SOLD BY ALLEN AND WEST, PATERNOSTER-ROW; AND GLENDINNING, CHARLES-STREET, HATTON-GARDEN.

1795.

ADDRESS

TO

CORRESPONDENTS, &c.

THE Editor *of the* Mathematical Repository *might be thought ungrateful in ushering his first Number to the world, without noticing the many ingenious Communications and Remarks that his numerous Correspondents have favoured him with for the Use of the Repository, and begs they would accept his warmest Expressions of Gratitude for their present Favours, relying on their Patronage and Support as a Reward for the laborious Task he has undertaken, and assures them that nothing but an ardent Desire of promoting the Study of the* Mathematics *could have engaged him in this Undertaking. Judging that other* Instruction *and* Information, *not yet laid before the Public, were wanting in many Departments of the Mathematics, which, in some Measure, has been overlooked, or at least but little noticed, therefore the Repository will be open to all, particularly where a superior Knowledge or Merit leads the Way; nor shall the Gleanings of diffident Merit or conscious Knowledge be overlooked: he therefore hopes that every Lover of the Mathematics will feel it their Duty to exert themselves in favour of a Work undertaken solely for their Use and Amusement.*

The

The Repository seems greatly called for at this Period, when Science and Literature are falling beneath the Influence of political Declamation, and when the Violence of Party is robbing Society of the Charms of social Conversation. If to divert the Mind from that which too surely fills the Breast with Rancour, and to direct it to Objects which humanize the Affections and give Charms to Sentiment be desirable, the Repository will have not inferior *Claims to public Approbation.*

Amongst other numerous Correspondents transmitted to him, he has to notice particularly those of Messrs. Burdon, Campbell, Gregory, Lowry, *and* Surtees. *Letters for the Use of No. II. as also Answers to the Questions proposed in* Prospectus, *must come to Hand before the 1st of January* 1796, *or they will be too late, addressed for the Editor of the Mathematical Repository, at Mr.* Glendinning's, *Bookseller, Charles Street, Hatton Garden, London.*

No. II. will be ready on the 26th of March 1796.

ERRATA.

Page 10, line 10, *for* AD by, *read* AD ; DB by.

THE

CONTENTS

OF THE

FIRST VOLUME.

XXII

*** Pages 265 to 274 are repeated.

THE MATHEMATICAL REPOSITORY.

ARTICLE I.

A DISSERTATION ON THE GEOMETRICAL ANALYSIS OF THE ANCIENTS.

IT is univerſally allowed that Mathematical Studies are attended with much pleaſure and delight, and that they are in a particular manner captivating and engaging to thoſe whoſe genius and bent of mind lie that way; and it is equally certain that they are profitable as well as pleaſant. From the proper cultivation of them many emoluments accrue to ſociety, and many improvements are made in civil life. But ſetting aſide theſe public advantages, for which the world muſt be indebted to men of ſuperior ability and exalted genius, qualified by nature to extend the

bounds

bounds of ſcience, and to derive practical improvements from their abſtract reaſonings; I would now only recommend theſe ſtudies, as uſeful to men of all ranks, profeſſions, and abilities, in their private capacities. They help greatly to fix the attention, to enlarge the underſtanding, to methodiſe our ideas, and to improve our reaſoning faculties, when we apply them to any other ſcience whatever. Great regard was had to theſe ſciences by the ancients in their education of children; and they ought certainly to have a principal ſhare in the education of gentlemen and ſcholars, if we would have them taught to think juſtly, or to reaſon well. Of all the branches of mathematical ſcience, none conduces more to the great ends here mentioned than Geometry. The ſimplicity of its firſt principles, the clearneſs and evidence of its demonſtrations, the admirable concatenation of its parts, and regular connexion of the propoſitions with one another, tend greatly to eſtabliſh a habit of cloſe thinking, and a methodical and juſt argumentation, when we apply ourſelves to any other ſubject whatſoever, phyſical or moral, œconomical or civil.

But I know not how it happens, this noble ſcience ſeems of late to have met with leſs regard than its dignity and uſefulneſs demand. Our ſchools and univerſities *, our philoſophical ſocieties, and philomaths of all degrees, ſeem to have been very aſſiduous of late in paying their devoirs to the younger ſiſter *Algebra*; and at the ſame time to have overlooked, and in a great meaſure neglected, thoſe native charms, that amiable ſimplicity, and thoſe more attractive excellencies, that peculiarly belong to the elder. I

* *The reader is referred to ſome late reſolutions taken in the univerſity of Cambridge.*

would

would not be thought any ways to diſparage the *algebraical* art, or to derogate from its merits; they are doubtleſs very great. All the branches of the mathematics are much indebted thereto; and even Geometry itſelf has, within this laſt century, been ſurpriſingly extended and improved thereby. But, notwithſtanding this, I muſt affirm that, for thoſe who apply to the mathematics only for the ends abovementioned, namely, to inure themſelves to a method of cloſe thinking and juſt reaſoning, *Geometry* is the proper field: and that thoſe who but moderately exerciſe themſelves therein, will ſooner attain their purpoſe than others that may go far greater lengths in the intricacies of *Algebra*, and the labyrinth of *Fluxions*.

It is but too common a practice with young ſtudents, after having gone through the Elements of *Euclid* in a curſory manner (or perhaps having ſubſtituted in their place ſome other *ſuperficial* and *leſs geometrical* elements) from henceforth to bid adieu to all demonſtrations ſtrictly geometrical, and to employ themſelves in the conſideration of algebraical equations, infinite ſeries, the algorithm of fluxions, the properties of curve lines, and other parts of the *higher* Geometry, when, perhaps, they are but very ſuperficially acquainted with the firſt elements of the *plane*. If they read a ſyſtem of conics, they will be ſure to make choice of one, where the demonſtrations are algebraical; and when they apply themſelves to the ſolution of geometrical problems, here again they will have recourſe to their Algebra, not at all appriſed, perhaps, of any other analyſis; being utterly unacquainted with the method of reſolution and compoſition ſo carefully obſerved by the ancient geometricians. Such ſtudents may become dextrous calculators, but ſuch a

method of proceeding tends very little towards bettering the judgment, or improving the invention.

The study of Geometry, I say, is the most proper for young men to pursue, in order to acquire a vigorous constitution of mind, and is as conducive thereto as exercise is towards procuring health and strength to the body. *Logical* precepts are useful, and indeed necessary for those that are engaged in public disputations, or controversial writings, in order to put to silence an obstinate adversary. But, 'in the search of 'truth, an imitation of the method of *geometers* will 'carry a man further than all the *dialectical* rules. 'Their *analysis* is the proper model we ought to form 'ourselves upon, and imitate in the regular disposi-'tion, and gradual progress of our enquiries *.'

We are told by Dr. *Pemberton* †, 'that Sir *Isaac* '*Newton* used to censure himself for not following 'the Ancients more closely than he did; and spoke 'with regret of his mistake, at the beginning of his 'mathematical studies, in applying himself to the 'works of *Descartes*, and other algebraical writers, 'before he had considered the Elements of *Euclid* 'with that attention so excellent a writer deserves. 'That he highly approved the laudable attempt of '*Hugo de Omerique* to restore the ancient *analysis*.' Now what the great Sir *Isaac Newton* so highly approved, it is the intention of this publication more particularly to specify and recommend. Little has yet been done toward the attainment of this laudable purpose of restoring the ancient *analysis*. The writer just mentioned is very little known in England. The

* Essay on the Usefulness of Mathematical Learning. Oxf. 1701.

† In the preface to his View, &c.

author

author of this ſmall tract is willing to contribute his mite, and very deſirous to revive a proper taſte for pure Geometry. He has annexed a collection of *Theorems*, and likewiſe a few *Problems*, to be ſolved by the *Geometrical Analyſis*; he has been more ſparing in the latter, becauſe plenty of them are continually propoſed in periodical publications. It is not pretended that they are new ones; but they are ſuch as rarely occur to them for whoſe uſe they are principally intended. Not above four or five of them, I believe, have ever appeared in Engliſh before; and they are all taken from authors which ſeldom fall into the hands of young men. They will ſerve, therefore, as proper exerciſes for young ſtudents to try their ſtrength upon.

But before they ſet themſelves to this work, I would recommend a very careful and reiterated peruſal of the Elements, and after that as diligent an application to that valuable remains of antiquity, the book of *Euclid's Data*, both which they will find moſt complete in Dr. *Robert Simſon*'s edition. When they have made themſelves perfect maſters of theſe, they may then betake themſelves to the ſolution of geometrical propoſitions by a geometrical analyſis; either that of the Ancients derived from the Data: or, if this ſhould be thought too tedious and troubleſome, they may abate ſomewhat of its rigour, and ſtill make uſe of a ſimilar method; but I would have them by no means content themſelves with algebraical reſolutions, even though they ſhould be able to derive conſtructions from thence, and alſo to demonſtrate ſynthetically the truth of the ſame. How they proceed with ſucceſs I ſhall endeavour briefly to explain.

Reſolution, then, or *Analyſis* is the method of proceeding from the thing *ſought* as *taken* for granted

through its consequences to something that is *really* granted; and *Composition* or *Synthesis* is a reverse method, wherein we lay that down first which was the last step of the *Analysis*, and tracing the steps of the *Analysis* back, making that antecedent here which was consequent there, till we arrive at the thing *sought*, which was put as granted in the first step of the Analysis.

Where we are to apply this method of *Resolution* to *Theorems*, we must first lay what is therein affirmed down as true, and then consider the necessary consequences flowing therefrom, deducing one consequence from another, till we arrive at last at some one, which is evidently true or evidently false, as may appear by an axiom, or an elementary proposition, or by what is called *Exposition*, i. e. the nature and structure of the figure. When the former is the case, the *Theorem* is true and may be demonstrated by the method of *Composition*; but when the latter is the case it is false, for all truths are consistent with each other. An example will clear this more than many words.

THEOREM.

The square of a line bisecting the verticle angle of any triangle, together with the rectangle under the segments of the base made thereby, is equal to the rectangle under the sides containing that angle.

ANALYSIS.

Suppose this to be true, viz. that $BD^2 + ADC = AB. BC$, (Fig. 1) and let a circle be circumscribed about the triangle, and BD produced to meet it in E,

E, and EC joined. Now $ADC = BDE$ by Euc. III. 35. Therefore $BD^2 + ADC = BD^2 + BDE = EBD$ by II. 3. Therefore, alſo, $AB.\ BC = EBD$. Now this we ſhall find to be true by the Elements, hence the theorem is alſo true; for the triangles ABD and EBC are ſimilar, having the angles at A and E equal as ſtanding on the ſame circumference, and the angles at B in each equal by *Expoſition*, therefore by VI. 4. $AB : BD :: BE : BC$ and by VI. 16. $AB.\ BC = EBD$.

SYNTHESIS.

$AB.\ BC = EBD$ (as proved in the Analyſis) $= BD^2 + BDE$ by II. 3. $= BD^2 + ADC$ by III. 35. *Q. E. D.*

It will be very proper for young ſtudents to endeavour to obtain a variety of demonſtrations of one and the ſame propoſition, deduced from different principles; hereby they will be able to diſcover which are the moſt ſimple and elegant, and greatly improve their taſte and judgment. An inſtance of the great fecundity of Geometry, in this reſpect, is given in the beginning of a periodical work publiſhed a few years ago under the title of the BRITISH ORACLE, where there are given fifteen demonſtrations of one and the ſame theorem, all independent of each other, and derived from very different principles and conſtructions. I ſhall now, therefore, proceed to give another demonſtration of the foregoing theorem, not as more ſimple than the preceding, but as leſs ſo, and further fetched. It was, however, given by an author of no ſmall note in the laſt century.

ANA-

ANALYSIS.

With one of the angular points at the baſe as center, and the adjacent ſegment as radius, deſcribe a circle, which let cut the adjacent ſide in F, and the ſame produced in G, and the biſecting line in K. AB : BC : : AD : DC by VI. 3. and AB : BC : : AB^2 : AB. BC, and AD : DC : : AD^2 : ADC by VI. 1. Therefore by V. 19. AD : DC : : $AB^2 - AD^2$: AB. BC—ADC. Now $AB^2 - AD^2 = AB^2 - AF^2$ = GBF by II. 6. = DBK by III. 36. and AB. BC—ADC = BD^2 by hypotheſis (for AB. BC is put = BD^2 + ADC.) therefore AD : DC : : DBK : BD^2. But DBK : BD^2 : : BK : BD by VI. 1. Therefore AD : DC : : BK : BD, which is true by Theorem LVIII. in the following collection.

SYNTHESIS.

AD : DC : : BK : BD (by the Theorem) : : DBK : DB^2 by VI. 1. Now DBK = GBF by III. 36. = $AB^2 - AF^2$ by II. 6. = $AB^2 - AD^2$. Therefore AD : DC : : $AB^2 - AD^2$: BD^2. Moreover AD : DC : : AD^2 : ADC by VI. 1. Therefore by V. 12. AD : DC : : AB^2 : BD^2 + ADC. Now AD : DC : : AB : BC by VI. 3. and AB : BC : : AB^2 : AB. BC by VI. 1. Therefore by equality AB^2 : BD^2 + ADC : : AB^2 : AB. BC, and by V. 9. BD^2 + ADC = AB. BC.——*Q. E. D.*

Here it is evident that this demonſtration falls ſhort of the preceding, becauſe it does not flow immediately from an elementary propoſition, but in order

der to its ratification the fore-cited Theorem LVIII. muſt be previouſly demonſtrated as a lemma.

When a *Problem* is propoſed to be ſolved, we muſt apply our method of *Reſolution* thus. We muſt conceive the thing required to be already done, and from this ſuppoſition we muſt reaſon, deducing one conſequence from another, and proceeding ſtep by ſtep, till we can arrive at ſomething that is granted, ſomething that may be affected by means of the poſtulates and elementary propoſitions, ſomething which (in the ſtyle of the Ancients) is given, or a *Datum*; which, if we can do, we ſhall then be able to form our *Syntheſis*, or *Compoſition*, by making the *Datum* we arrived at in the laſt ſtep of our analyſis, the firſt ſtep or foundation of our ſyntheſis; and then reaſoning in a retrograde order, and taking the ſame ſteps back again, we ſhall deduce one conſequence from another, till we arrive at the original *Quæſitum*, or thing required to be done in the problem propoſed, which was the firſt thing laid down and ſuppoſed in our analyſis.

Take the following example, being the 155th propoſition of *Pappus*'s VIIth Book.

PROBLEM.

It is required in a given ſegment of a circle from the extremes of the baſe A and B, (Fig. 2.) to draw two lines, AC and BC, meeting at a point C in the circumference, and ſuch that they ſhall have a given ratio to each other, viz. that of F to G.

ANALYSIS.

Suppoſe the thing done, and that the point C is found; then, by way of preparation or conſtruction, or

or ſomething to found our analyſis upon, let us ſuppoſe that a tangent to the ſegment at the point C is drawn, which meets AB produced in D. Now by hypotheſis AC : CB : : F : G, alſo $AC^2 : CB^2$: : AD : DB, which is thus proved.

Since DC touches the circle, and BC cuts it, the angle DCB = BAC by III. 32. Alſo the angle D is common to both the triangles CDB and CDA, therefore they are ſimilar, and by VI. 4. AD : DC : : DC : DB, hence $AD^2 : DC^2$: : AD by VI. 20. cor. But alſo by VI. 4. AD : AC : : DC : CB, and by permutation AD : DC : : AC : CB, or AD^2 : DC^2 : : $AC^2 : CB^2$, therefore by equality AC^2 : CB^2 : : AD : DB.

But the ratio of $AC^2 : CB^2$ is given (by Prop. LVII. in Dr. Simſon's edition of the *Data* *) becauſe the ratio of AC : CB is given, therefore, alſo, that of AD : AB. Now, ſince the ratio of AD : DB is given, therefore, alſo, by *Data* VI. that of AD : AB, and hence by *Data* II. AD is given in magnitude.

And here the analyſis properly ends. For it having been ſhewn that AD is given, or that a point D may be found in AB produced ſuch, that from it a tangent being drawn to the circumference, the point of contact will be the point ſought; we may now begin our compoſition, or ſynthetical demonſtration, which we muſt do by finding the point D, or laying down the line AD, which we affirmed to be given in the laſt ſtep of our analyſis.

* *Dr. Simſon has altered the order of the propoſitions of this book, but by marginal figures referred to the original order in the Greek text.*

SYNTHESIS.

SYNTHESIS.

Construction. Make as $F^2 : G^2 :: AD : DB$ (which may be done, ſince AB is given, by making it as $F^2 - G^2 : G^2 :: AB : DB$, and then by compoſition it will be as $F^2 : G^2 :: AD : DB$) and then from the point D thus found draw a tangent to the circle, and from the point of contact C drawing CA and CB the thing is done.

Demonſtration. Since by *conſtruction* $F^2 : G^2 :: AD : DB$, and alſo $AD : DB :: AC^2 : BC^2$ (which has been already demonſtrated in the analyſis, and may be here proved in the ſame manner.) Therefore $F^2 : G^2 :: AC^2 : BC^2$, and conſequently $F : G :: AC : BC$. *Q. E. D.*

Here we ſee an inſtance of the method of *Reſolution* and *Compoſition*, as it was practiſed by the Ancients, for the ſolution here given is that of *Pappus Alexandrinus*. But becauſe the method of referring and reducing every thing to the *Data*, and conſtantly quoting the ſame, may appear to many to be very tedious and troubleſome: and, indeed, it is unneceſſary to thoſe who have already made themſelves maſters of the ſubſtance of this valuable book of *Euclid*, and have, by practice and experience, acquired a facility of reaſoning in ſuch matters; I ſhall therefore now ſhew how we may abate ſomething of the rigour and ſtrict form of the ancient method of demonſtration, without diminiſhing any part of its admirable perſpicuity and elegance. And this I ſhall do by inſtancing in another ſolution or two of the ſame problem.

But before I do this, it may be proper to take notice that, in this buſineſs of the reſolution of problems, every thing cannot be brought within ſtrict rules,

rules, nor any infallible directions given whereby a man may be enabled to succeed in all possible cases; but that there is need of a previous preparation, a kind of mental contrivance and construction, in order to form a connexion between the *Data* and *Quæsita*, which must be left to every one's own sagacity to find out. And it is on this very account chiefly that I recommend the exercise and employment of solving *Geometrical* Problems as a means to help our invention, and to improve and strengthen our reasoning powers, when we employ them on any other subject of a different nature. And here it is that the geometrician shews his taste and acuteness in choosing the proper foundation whereon to build the most elegant construction and demonstration; and the mathematical reader his judgment in being able, amidst a variety, to distinguish the same. However some people may be apt to ridicule the notion of taste, when applied to these subjects, yet I do maintain that the word and thing signified thereby is very properly applicable thereto: and I believe all true mathematicians will bear me out, when I affirm that a man may shew as much taste in being able to distinguish between the different degrees of elegance in mathematical demonstrations, as a student in the *Belles Lettres*, or a professed critic can by his right relish for the *sublime* or *pathetic* in poetry or oratory.

But to return: before we begin our analysis, I say there is most commonly some preparation necessary, in order to form a connexion between the *Data* and *Quæsita*, which cannot fall within any rules, but is various according to the various nature of the problems proposed. Right lines must be drawn in particular directions, or of particular magnitudes, bisecting, perhaps, a given angle, or perpendicular to a given line; tangents from a given point to a given curve;

curve; circles muſt be deſcribed from a given centre with a given radius; or touching given lines or other given circles; or ſuch-like other operations. Whoever is converſant with the works of *Archimides*, *Apollonius*, or *Pappus*, knows very well that they all found their analyſis upon ſuch-like previous operations. Now the great ſkill of the analyiſt conſiſts in diſcovering the moſt proper *effections* whereon to found his analyſis: and young ſtudents ought to exerciſe their abilities and improve their ſagacity therein; for the ſame problem may frequently be conſtructed many different ways, and many different demonſtrations given of the ſame, which I ſhall now exemplify by giving two more different ſolutions of the aforeſaid problem.

ANALYSIS.

Let us ſuppoſe then again that the thing is done, *i. e.* AC : CB : : F : G (fig. 2) and let the baſe of the ſegment be cut in the ſame ratio in the point E; then EC being drawn will biſect the angle ACB by VI. 3. conſequently if the circle be completed, and CE produced to meet it in K, the remaining circumference will alſo be biſected in K; therefore the point K, as well as E, being given, the point C muſt alſo be given.

SYNTHESIS.

Conſtruction. Let the given baſe of the ſegment AB be cut in the point E in the aſſigned ratio of F : G by VI. 10. and complete the circle by III. 25. biſect the remaining circumference in K by III. 30. join KE and continue it to meet the circumference in C, and drawing CA, CB, the thing is done.

Demonstration. Since the arc KC = the arc KB, the angle ACK = angle BCK by III. 27. therefore AC : CB : : AE : EB by VI. 3. but AE : EB : : F : G by *construction*; therefore AC : CB : : F : G. *Q. E. D.*

Again: this problem may be ſolved in another manner, by conſidering it in a little different light; for it is the ſame thing required, as having the baſe of a triangle given, together with the vertical angle and the ratio of the legs, to find the triangle.

Let P = the given angle, or, in other words, the angle in the given ſegment. Set off upon the legs of the angle P, PM and PQ in the aſſigned ratio of F : G, and join MQ; then upon the given baſe AB make a triangle ACB ſimilar to MPQ by VI. 18. and the thing is done; and the demonſtration follows immediately from VI. 4.

Having now given a ſpecimen of what kind of ſolutions I would have the young ſtudent endeavour after, I have nothing further to add, but to adviſe him, when he undertakes a propoſition, not to be diſcouraged by one or two fruitleſs attempts, and thereupon throw it aſide, or ranſack books for the ſolution; but if he would profit by this exerciſe, let him perſevere, encouraged by the words of *Syrus* in the play:

Nil tam difficile eſt quin quærendo inveſtigari poteſt.

ARTICL

ARTICLE II.

Containing the first Eight Propositions of STEWART'S GENERAL THEOREMS, *being all that the Author has demonstrated.*

PROPOSITION I. Fig. 3.

IF from A the vertex of any triangle ABC there be drawn AD to any point D in the base, and DE, DF be drawn parallel to AC, AB meeting AB, AC in E, F, the sum of the rectangles BAE, CAF will be equal to the square of AD together with the rectangle BDC.

About the triangle ABC let there be a circle described, and let AD meet the circle in G; join BG, CG, and from the point E draw EH making the angle AHE equal to the angle ABG, and produce AB to any point K.

Because the angles AHE, ABG are equal, the points E, B, G, H, are in a circle, therefore the rectangle BAE is equal to the rectangle GAH. The angle EHD will also be equal to the angle GBK, that is, to the angle ACG. And because AC, DE are parallel, the angles EDH, GAC will be equal; therefore the triangles EDH, GAC will be similar; and therefore AC will be to AG as DH to DE; therefore the rectangle contained by AC, DE, that is, the rectangle CAF, is equal to the rectangle contained by AG, DH. But because the rectangle BAE is equal to the rectangle GAH, and likewise the rect-

ang'e CAF equal to the rectangle contained by AG, DH, the ſum of the rectangles BAE, CAF will be equal to the rectangle GAD, that is, equal to the rectangle ADG together with the ſquare of AD. But the rectangle ADG is (III. 35.) equal to the rectangle BDC; therefore the ſum of the rectangles BAE, CAF is equal to the ſquare of AD together with the rectangle BDC. *Q. E. D.*

PROPOSITION II. Fig. 4, 5.

In the right line AB take any point C between the points A, B; and from the points A, B, C let there be drawn right lines to any point D; the ſquare of AD together with the ſpace to which the ſquare of BD has the ſame ratio that BC has to CA, will be equal to the rectangle BAC together with the ſpace to which the ſquare of CD has the ſame ratio that BC has to BA.

Firſt. When the point D (fig. 4.) is not in the line AB.

Draw AE, DF parallel to CD, AB meeting BD, AE in E, F.

Becauſe the ſquare of BD is to the rectangle BDE as BD to DE, that is, as BC to CA, the rectangle BDE will be the ſpace to which the ſquare of BD has the ſame ratio that BC has to CA; and becauſe the ſquare of AF, that is, the ſquare of CD, is to the rectangle EAF as AF to AE, that is, as BD to BE, or BC to BA, the rectangle EAF will be the ſpace to which the ſquare of CD has the ſame ratio that BC has to BA. But (Prop. I.) the ſquare of AD together with the rectangle BDE, is equal to the rectangle BAC together with the rectangle EAF; therefore the ſquare of AD together with the ſpace to

to which the ſquare of BD has the ſame ratio that BC has to CA, is equal to the rectangle BAC together with the ſpace to which the ſquare of CD has the ſame ratio that BC has to BA. *Q. E. D.*

Second. When the point D (fig. 5.) is in the line AB.

Draw CE perpendicular to AB, and let CE be equal to AC; join AE, BE; draw BF parallel to CE meeting AE in F; and draw DG parallel to CE or BF meeting AE, BE in G, H; and join GC, HC.

Becauſe AC is equal to CE, AD will be equal to DG; therefore the ſquare of AD will be equal to twice the triangle ADG; and becauſe the ſquare of BD is to the rectangle BDH, that is, twice the triangle BDH, as BC to CE, or CA, twice the triangle BDH will be the ſpace to which the ſquare of BD has the ſame ratio that BC has to CA. Again, becauſe AC, CE are equal, the rectangle BAC will be equal to twice the triangle AEB; and becauſe EG is to EF, that is, CD to CB as GH to BF, or AB, CD will be to GH as BC to AB; therefore the ſquare of CD will be to the rectangle contained by CD, GH, that is, twice the triangle GCH, or GEH, as BC to AB; therefore twice the triangle GEH will be the ſpace to which the ſquare of CD has the ſame ratio that BC has to AB. But it is evident, that twice the ſum of the triangles ADG, BDH is equal to twice the ſum of the triangles AEB, GEH; therefore the ſquare of AD together with the ſpace to which the the ſquare of BD has the ſame ratio that BC has to CA, is equal to the rectangle BAC together with the ſpace to which the ſquare of CD has the ſame ratio that BC has to AB. *Q. E. D.*

Corollary. If from the vertex of any triangle there be drawn a line biſecting the baſe, the

ſum of the ſquares of the ſides of the triangle will be equal to twice the ſquare of the line biſecting the baſe together with the ſum of the ſquares of the ſegments of the baſe.

PROPOSITION III.

THEOREM I. FIG. 6.

Let there be any regular figure ABC circumſcribed about a circle, and from any point D within the figure let there be drawn DE, DF, DG perpendicular to the ſides of the figure; the ſum of the perpendiculars DE, DF, DG will be equal to the multiple of the ſemidiameter of the circle by the number of the ſides of the figure.

Join DA, DB, DC. The figure ABC will be divided into as many triangles as there are ſides in the figure; and becauſe every one of the triangles is equal to half the rectangle contained by the baſe and the perpendicular drawn from the vertex to the baſe, and all the baſes are equal, becauſe the figure is regular; therefore the ſum of all the triangles will be equal to half the rectangle contained by the ſum of the perpendiculars and one of the ſides of the figure; and therefore twice the figure will be equal to the rectangle contained by the ſum of the perpendiculars and one of the ſides of the figure. But the rectangle contained by the ſemidiameter of the circle and the ſum of the ſides of the figure, is equal to twice the figure; therefore the rectangle contained by the ſum of the perpendiculars DE, DF, DG, and one of the ſides of the figure, is equal to the rectangle contained by the ſemidiameter of the circle and the ſum of the ſides of the figure; and therefore the ſum of

 the

the perpendiculars DE, DF, DG will be to the ſemidiameter of the circle, as the ſum of the ſides of the figure to one of the ſides of the figure, that is, as the number of the ſides of the figure to one; therefore the ſum of the perpendiculars DE, DF, DG is equal to the multiple of the ſemidiameter of the circle by the number of the ſides of the figure. *Q. E. D.*

LEMMA II. Fig. 7.

Let there be any circle ABC, and let AD be a tangent to the circle in the point A; from the point A let there be drawn AB to any point B in the circle, and let BD be perpendicular to AD; the ſquare of AB will be equal to the rectangle contained by BD and the diameter.

Let AC be the diameter of the circle, and join BC, becauſe the angles ACB, BAD are (III. 32.) equal, and the angles ABC, BDA likewiſe equal, becauſe both right, the triangles ABC, ADB will be ſimilar; therefore AC will be to AB as AB to BD; therefore the ſquare of AB is equal to the rectangle contained by BD, AC. *Q. E. D.*

PROPOSITION IV.

THEOREM II. FIG. 8, 9.

Let the circumference of a circle be divided into any number of equal parts in the points A, B, C, &c. and from the points A, B, C, &c. let there be drawn right lines to any point D, the ſum of the ſquares of AD, BD, CD, &c. will be equal to the multiple of the ſquare of the line drawn from the center of the circle

circle to the point D by the number of the points A, B, C, &c. together with the multiple of the ſquare of the ſemidiameter by the ſame number.

Firſt. When the point D (fig. 8.) is in the circumference of the circle, it is to be ſhewn that the ſum of the ſquares of AD, BD, CD, &c. is equal to twice the multiple of the ſquare of the ſemidiameter by the number of the points A, B, C, &c.

Let there be a regular figure circumſcribed about the circle, touching the circle in the points A, B, C, &c. and draw DE, DF, DG perpendicular to the ſides of the figure; becauſe the ſquare of AD is (*Lem.* 1.) equal to the rectangle contained by DE and the diameter, and likewiſe the ſquare of BD, equal to the rectangle contained by DF and the diameter, and ſo on; it is evident that the ſum of the ſquares of AD, BD, CD, &c. will be equal to the rectangle contained by the ſum of the perpendiculars DE, DF, DG, &c. and the diameter. But becauſe (Pro. III.) the ſum of the perpendiculars DE, DF, DG, &c. is equal to the multiple of the ſemidiameter by the number of the ſides of the circumſcribed figure, that is, by the number of the points A, B, C, &c. the rectangle contained by the ſum of the perpendiculars DE, DF, DG, &c. and the diameter, will be equal to twice the multiple of the ſquare of the ſemidiameter by the number of the points A, B, C, &c. therefore the ſum of the ſquares of AD, BD, CD, &c. will be equal to twice the multiple of the ſquare of the ſemidiameter by the number of the points A, B, C, &c. *Q. E. D.*

Second. When the point D (fig. 9.) is not in the circumference of the circle, it is to be ſhewn that the ſum of the ſquares of AD, BD, CD, &c. is equal to the multiple of the ſquare of the line drawn from the center of the circle to the point D by the number

number of the points A, B, C, &c. together with the multiple of the ſquare of the ſemidiameter by the ſame number.

Let E be the center of the circle, and join DE; let DE meet the circle in the point F on the other ſide of the center E, and join AE, BE, CE, &c. AF, BF, CF, &c. the ſquare of AD together with the ſpace to which the ſquare of AF has the ſame ratio that EF has to ED, will (Pro. II.) be equal to the rectangle EDF together with the ſpace to which the ſquare of AE, or EF, has the ſame ratio that EF has to FD, that is, together with the rectangle EFD; and therefore the ſquare of AD together with the ſpace to which the ſquare of AF has the ſame ratio that EF has to ED, will be equal to the ſquare of DF. The ſame way it is ſhewn that the ſquare of BD together with the ſpace to which the ſquare of BF has the ſame ratio that EF has to ED, is equal to the ſquare of DF, and ſo on; therefore the ſum of the ſquares of AD, BD, CD, &c. together with the ſpace to which the ſum of the ſquares of AF, BF, CF, &c. has the ſame ratio that EF has to ED, will be equal to the multiple of the ſquare of DF by the number of the points A, B, C, &c. but becauſe the ſum of the ſquares of AF, BF, CF, &c. is equal (by the firſt part of this) to twice the multiple of the ſquare of EF by the number of the points A, B, C, &c. the ſpace to which the ſum of the ſquares of AF, BF, CF, &c. has the ſame ratio that EF has to ED, will be equal to twice the multiple of the rectangle FED by the number of the points A, B, C, &c. therefore the ſum of the ſquares of AD, BD, CD, &c. together with twice the multiple of the rectangle FED by the number of the points A, B, C, &c. is equal to the multiple of the ſquare of DF by the ſame number; and therefore the

the ſum of the ſquares of AD, BD, CD, &c. is equal to the multiple of the ſum of the ſquares of DE, EF by the number of the points A, B, C, &c. *Q. E. D.*

COR. I. Let there be two circles having the ſame center, and let the circumference of one of the circles be divided into any number of equal parts, and from the points of diviſion let there be drawn right lines to any point in the circumference of the other, the ſum of the ſquares of theſe lines will always be the ſame.

COR. II. Let there be two regular figures inſcribed in a circle, and from all the angles of both figures let there be drawn right lines to any point, the ſum of the ſquares of the lines drawn from the angles of the one, will be to the ſum of the ſquares of the lines from the angles of the other, as the number of the ſides of the one to the number of the ſides of the other.

LEMMA II. Fig. 10, 11.

Let there be any number of right lines AB, AC, AD, AE, &c. interſecting each other in the point A, and making all the angles about the point A equal; let there be any circle paſſing through the point A, the circumference of the circle will be divided by the lines interſecting each other in the point A into as many equal parts as there are lines.

Firſt. When the circle does not touch any of the lines interſecting each other in the point A (fig. 10.).

Let AB, AC, AD, AE, &c. meet the circle in B, C, D, E, &c. becauſe the angles BAC, CAD, DAE, &c. are equal, the ſegments BC, CD, DE, &c. will be equal. Let BE be the ſegment in which the point A is;

A is; draw BD, ED to any point D in the circle, the angle BDE will be equal to the angle adjacent to the angle BAE, that is, to the angle BAF, or BAC; therefore the ſegment BAE is equal to the ſegment BC.

Second. When the circle touches one of the lines interſecting each other in the point A (fig. 11.) let it touch AB, and let AC, AD, AE meet the circle in C, D, E; becauſe the angle CAD is equal to the angle DAE, CD will be equal to DE, &c. join CD; and becauſe the angle ADC is equal to the angle CAB, that is, to the angle CAD, or DAE, the ſegment AC will be equal to the ſegment CD, or DE. The ſame way it is ſhewn that the ſegment AE is equal to the ſegment DE, or DC; therefore the lemma is evident. *Q. E. D.*

PROPOSITION V.

THEOREM III. FIG. 12, 13.

Let there be any regular figure circumſcribed about a circle, and from any point let there be drawn perpendiculars to the ſides of the figure; and likewiſe a right line to the center of the circle, twice the ſum of the ſquares of the perpendiculars to the ſides of the figure, will be equal to the multiple of the ſquare of the line drawn to the center by the number of the ſides of the figure, together with twice the multiple of the ſquare of the ſemidiameter by the ſame number.

Firſt. When the number of the ſides of the figure circumſcribed about the circle is even (fig. 12.).

Let ABCDEF, &c. be any regular figure of an even number of ſides circumſcribed about a circle, and from any point G let there be drawn GH, GK, GL, GM, GN, GO perpendicular to the ſides of the figure, and let a be the center of the circle, and join Ga, twice the ſum of the ſquares GH, GK, GL,

GL, GM, GN, GO, &c. will be equal to the multiple of the ſquare of Ga by the number of the ſides of the figure, together with twice the multiple of the ſquare of the ſemidiameter of the circle by the ſame number.

Let the circumſcribed figure touch the circle in P, Q, R, S, T, V, &c. and join GP, GQ, GR, GS, GT, GV, &c. join aP, aQ, aR, aS, aT, aV, &c. and draw GX, GY, GZ, &c. perpendicular to aP, aQ, aR, &c.

Becauſe the number of the ſides of the circumſcribed figure is even, it is plain, aP, aQ, aR, &c. will paſs through the oppoſite points of contact, that is, through the points S, T, V; and therefore the number of lines interſecting each other in the point a will be half the number of the ſides of the figure, and all the angles round the point a will be equal. Becauſe the ſum of the ſquares GX, GH is equal to the ſquare of GP, and the ſum of the ſquares of GK, GY equal to the ſquare of GQ, and ſo on; it is evident, that the ſum of the ſquares of GH, GK, GL, GM, GN, GO, &c. together with twice the ſum of the ſquares of GX, GY, GZ, &c. is equal to the ſum of the ſquares of GP, GQ, GR, GS, GT, GV, &c. that is, (Prop. IV.), equal to the multiple of the ſquare of Ga by the number of the ſides of figure, together with the multiple of the ſquare of the ſemidiameter of the circle by the ſame number. Therefore twice the ſum of the ſquares of GH, GK, GL, GM, GN, GO, &c. together with four times the ſum of the ſquares of GX, GY, GZ, &c. will be equal to twice the multiple of the ſquare of Ga by the number of the ſides of the figure, together with twice the multiple of the ſquare of the ſemidiameter of the circle by the ſame number. Again, Becauſe the angles GXa, GZa, GYa are right,

right, the points X, Y, Z will be in the circumference of the circle whose diameter is G*a*; and becaufe the circle paffes through the point *a*, the circumference will be divided into equal parts in the points X, Y, Z, as many in number as there are right lines *a*P, *a*Q, *a*R, &c. (*Lem.* II.). Bifect G*a* in *b*; the fum of the fquares GX, GY, GZ, &c. will (Prop. IV.) be equal to twice the multiple of the fquare of G*b* by the number of the lines *a*P, *a*Q, *a*R, &c. that is, (becaufe the number of the lines *a*P, *a*Q, *a*R, &c. is equal to half the number of the fides of the figure), equal to the multiple of the fquare of G*b* by the number of the fides of the circumfcribed figure; and therefore four times the fum of the fquares of GX, GY, GZ, &c. will be equal to the multiple of the fquare of *a*G by the number of the fides of the figure; therefore twice the fum of the fquares of GH, GK, GL, GM, GN, GO, &c. together with the multiple of the fquare of G*a* by the number of the fides of the circumfcribed figure, will be equal to twice the multiple of the fquare of G*a* by the number of the fides of the figure, together with twice the multiple of the fquare of the femidiameter by the fame number; and therefore twice the fum of the fquares of GH, GK, GL, GM, GN, GO, &c. will be equal to the multiple of the fquare of G*a* by the number of the fides of the figure, together with twice the multiple of the fquare of the femidiameter by the fame number.

Second. When the number of the fides of the figure circumfcribed about the circle is odd (fig. 13.).

Let ABCDE, &c. be any regular figure of an odd number of fides circumfcribed about a circle, and from any point F let there be drawn FG, FH, FK, FL, FM, &c. perpendicular to the fides of the figure, and let *a* be the center of the circle, and join F*a*, twice

the ſum of the ſquares of FG, FH, FK, FL, FM, &c. will be equal to the multiple of the ſquare of F*a* by the number of the ſides of the figure together with twice the multiple of the ſquare of the ſemidiameter of the circle by the ſame number.

Let the circumſcribed figure touch the circle in the points N, O, P, Q, R, &c. and join FN, FO, FP, FQ, FR, &c. join *a*N, *a*O, *a*P, *a*Q, *a*R, &c. and draw FS, FT, FV, FX, FY, &c. perpendicular to *a*N, *a*O, *a*P, *a*Q, *a*R, &c. Becauſe the ſum of the ſquares of FG, FS is equal to the ſquare of FN, and the ſum of the ſquares of FH, FT equal to the ſquare of FO, and ſo on, it is evident that the ſum of the ſquares of FG, FH, FK, FL, FM, &c. together with the ſum of the ſquares of FS, FT, FV, FX, FY, &c. is equal to the ſum of the ſquares of FN, FO, FP, FQ, FR, &c. that is, (Prop. IV.) equal to the multiple of the ſquare of F*a* by the number of the ſides of the figure together with the multiple of the ſquare of the ſemidiameter by the ſame number; therefore twice the ſum of the ſquares of FG, FH, FK, FL, FM, &c. together with twice the ſum of the ſquares of FS, FT, FV, FX, FY, &c. is equal to twice the multiple of the ſquare of F*a* by the number of the ſides of the figure together with twice the multiple of the ſquare of the ſemidiameter of the circle by the ſame number. Again, Becauſe the angles FS*a*, FT*a*, FV*a*, FX*a*, FY*a*, &c. are right, the points S, T, V, X, Y, &c. will be in the circumference of a circle whoſe diameter is F*a*; and becauſe the circle paſſes through the point *a*, and the lines *a*N, *a*O, *a*P, *a*Q, *a*R, &c. make all the angles round the point *a* equal, the circumference of the circle will be divided into equal parts in the points S, T, V, X, Y, &c. as many in number as there are right lines *a*N, *a*O, *a*P, *a*Q, *a*R,

*a*R, &c. (*Lem.* II.). Biſect F*a* in *b*; the ſum of the ſquares of FS, FT, FV, FX, FY, &c. will be equal to twice the multiple of the ſquare of F*b* by the number of the lines *a*N, *a*O, *a*P, *a*Q, *a*R, &c. (Prop. IV.) that is, by the number of the ſides of the circumſcribed figure; and therefore twice the ſum of the ſquares of FS, FT, FV, FX, FY, &c. will be equal to the multiple of the ſquare of F*a* by the number of the ſides of the figure. Therefore twice the ſum of the ſquares of FG, FH, FK, FL, FM, &c. together with the multiple of the ſquare of F*a* by the number of the ſides of the figure, will be equal to twice the multiple of the ſquare of F*a* by the number of the ſides of the figure together with twice the multiple of the ſquare of the ſemidiameter of the circle by the ſame number. And therefore twice the ſum of the ſquares of FG, FH, FK, FL, FM, &c. will be equal to the multiple of the ſquare of F*a* by the number of the ſides of the figure together with twice the multiple of the ſquare of the ſemidiameter of the circle by the ſame number. *Q. E. D.*

Cor. I. Let there be any regular figure circumſcribed about a circle, and from any point in the circumference of the circle let there be drawn perpendiculars to the ſides of the figure; twice the ſum of the ſquares of the perpendiculars will be equal to thrice the multiple of the ſquare of the ſemidiameter of the circle by the number of the ſides of the figure.

Cor. II. Let there be two circles having the ſame center, and from any point of the circumference of the one let there be drawn perpendiculars to the ſides of any regular figure circumſcribed about the other; the ſum of the ſquares of theſe perpendiculars will always be the ſame.

Cor. III. Let there be two regular figures circumscribed about a circle, and from any point let there be drawn perpendiculars to the sides of both figures; the sum of the squares of the perpendiculars drawn to the sides of the one, will be to the sum of the squares of the perpendiculars drawn to the sides of the other, as the number of the sides of the one to the number of the sides of the other.

PROPOSITION VI. Fig. 14, 15.

Let A, B be two points in the semidiameter of a circle whose center is C, and let the rectangle ACB be equal to the square of the semidiameter; bisect AB in D, and draw DE perpendicular to AB; from the point A draw AF to any point F in the circle, and draw FE perpendicular to DE; the square of AF will be equal to twice the rectangle contained by AC, FE.

Let CG be equal to AC, and join GF; let EF meet the circle in H, and join AH, GH, AE, CE, CF, CH; and let CE meet the circle in K, L.

The square of CD is equal to the rectangle ACB together with the square of AD, that is, equal to the square of the semidiameter together with the square of AD. Add the square of DE to both; and the square of CE will be equal to the square of the semidiameter together with the square of AE. Take away the square of the semidiameter from both, and the square of AE will be equal to the rectangle KEL, that is, equal to the rectangle FEH; and therefore FE is to AE, as AE to EH: therefore the triangles AEF, AHE are similar, and the angle EAF will be equal to the angle AHE, that is, equal to the angle HAG. Again, Because the angle ACF is equal to the

the angle CFH, that is, equal to the angle CHF, or GCH, the angle ACH will be equal to the angle GCF; and becaufe AC, CH are equal to GC, CF, the triangles ACH, GCF will be every way equal, and the angle GCF will be equal to the angle HAG, that is, equal to the angle EAF; and becaufe the angles EFA, FAG are equal, the triangles AEF, FAG will be fimilar; and therefore EF will be to AF, as AF to AG; therefore the fquare of AF is equal to the rectangle contained by EF, AG, that is, equal to twice the rectangle contained by EF, AC. *Q. E. D.*

PROPOSITION VII.

THEOREM IV. FIG. 16, 17.

Let there be any circle whofe center is A, and let BCD be a fegment of the circle, and BD the chord of the fegment; about the fegment let there be any equilateral figure circumfcribed touching the circle in the points E, F, G, &c. and let the two fides of the figure next to BD meet BD in H, K; bifect the fegment BCD in F, and join AF; in AF take the point L on the fame fide the center A with the point F, and let the fum of fides of the figure circumfcribed about the fegment be to HK as the femidiameter to AL; draw LM perpendicular to AL meeting the circle in M. If from the points E, F, G, &c. the points of contact of the circumfcribed figure, and the point L, there be drawn right lines to any point N, the fum of the fquares of EN, FN, GN, &c. will be equal to the multiple of the fum of the fquares of LM, LN by the number of the fides of the figure.

First. When the point N is in the circumference of the circle (fig. 16.).

In AF take the point O, and let the rectangle LAO be equal to the square of the semidiameter of the circle, and let OP be perpendicular to AF; draw NP perpendicular to OP; bisect LO in Q, and let QR parallel to OP meet NP in R; let NP, AO meet BD in S, T, and join AH, AK, NH, NK; and join likewise AM; and draw NV, NX, NY, &c. perpendicular to the sides of the figure meeting the sides of the figure in V, X, Y, &c. because the rectangle LAO is equal to the square of the semidiameter of the circle, that is, equal to the square of AF; AO will be to AF, as AF to AL, that is, as the sum of the sides of the figure circumscribed about the segment to HK; and therefore the rectangle contained by AO, HK, will be equal to the rectangle contained by AF, and the sum of the sides of the figure, that is, will be equal to twice the figure AHEFGKA; and because the rectangle contained AT, HK is equal to twice the triangle AHK, the rectangle contained by OT, HK will be equal to twice the figure HEFGKH, that is, the rectangle contained by PS, HK will be equal to the figure HEFGKH. Again, Because the rectangle contained by NS, HK is equal to twice the triangle NHK, the rectangle contained by NP, HK will be equal to twice the figure NHEFGKN. But the rectangle contained by the sum of the perpendiculars NV, NX, NY, &c. and one of the sides of the figure, is equal to twice the figure NHEFGKN; therefore the rectangle contained by NP, HK, is equal to the rectangle contained by the sum of NV, NX, NY, &c. and one of the sides of the figure; and therefore NP will be to one of the sides of the figure, as the sum of

of the perpendiculars NV, NX, NY to HK, that is, the multiple of NP by the number of the ſides of the figure, will be to the ſum of the ſides of the figure, as the ſum of the perpendiculars NV, NX, NY, &c. to HK; therefore the multiple of NP by the number of the ſides of the figure, will be to the ſum of the perpendiculars NV, NX, NY, as the ſum of the ſides of the figure to HK, that is, as AF, to AL, or twice AF to twice AL; therefore twice the multiple of the rectangle contained by NP, AL by the number of the ſides of the figure, is equal to the rectangle contained by the ſum of the perpendiculars NV, NX, NY, &c. and twice AF. But becauſe (*Lem.* 1.) the ſquare of NE is equal to the rectangle contained by NV and twice AF, and the ſquare of NF equal to the rectangle contained by NX and twice AF, and the ſquare of NG equal to the rectangle contained by NY and twice AF, and ſo on; the ſum of the ſquares of NE, NF, NG, &c. will be equal to the rectangle contained by the ſum of the perpendiculars NV, NX, NY, &c. and twice AF, that is, will be equal to twice the multiple of the rectangle contained by NP, AL by the number of the ſides of the figure. Again, Becauſe the rectangle OAL is equal to the ſquare of AM, the rectangle OLA will be equal to the ſquare of LM, that is, twice the rectangle contained by PR, AL will be equal to the ſquare of LM. And becauſe (Prop. VI.) twice the rectangle contained by NR, AL is equal to the ſquare of LN, twice the rectangle contained by NP, AL will be equal to the ſum of the ſquares of LM, LN; therefore twice the multiple of the rectangle contained by NP, AL by the number of the ſides of the figure, will be equal to the multiple of the ſum of the ſquares of LM, LN by the ſame number; and therefore the ſum of the ſquares of NE, NF, NG,

NG, &c. will be equal to the multiple of the ſum of the ſquares of LM, LN by the number of the ſides of the figure. *Q. E. D.*

Second. When the point N is not in the circumference of the circle (fig. 17.).

Join NA, and let NA meet the circle in the point O on the other ſide the center A with the point N; and join EO, FO, GO, &c. LO; join likewiſe EA, FA, GA, &c. and join AM; let LO meet the circle in P, and draw NQ parallel to AL meeting OL in Q. Becauſe (Prop. II.) the ſquare of EN, together with the ſpace to which the ſquare of EO has the ſame ratio that OA has to AN, is equal to the rectangle ANO together with the ſpace to which the ſquare of AE has the ſame ratio that AO has to ON, and the ſquare of AE is equal to the ſum of the ſquares of AL, LM; the ſquare of EN together with the ſpace to which the ſquare of EO has the ſame ratio that AO has to AN, will be equal to the rectangle ANO together with the ſpace to which the ſquare of AL has the ſame ratio that AO has to ON together with the ſpace to which the ſquare of LM has the ſame ratio. But (Prop. II.) the rectangle ANO together with the ſpace to which the ſquare of AL has the ſame ratio that AO has to NO, is equal to the ſquare of NL together with the ſpace to which the ſquare of LO has the ſame ratio that OA has to AN; therefore the ſquare of EN together with the ſpace to which the ſquare of EO has the ſame ratio that OA has to AN, is equal to the ſquare of NL together with the ſpace to which the ſquare of LO has the ſame ratio that OA has to AN together with the ſpace to which the ſquare of LM has the ſame ratio that AO has to ON; becauſe the ſquare of LO is to the rectangle OLQ as OL to LQ, that is, as OA to AN, the rectangle OLQ will be

be the ſpace to which the ſquare of OL has the ſame ratio that OA has to AN. And becauſe the rectangle OLP is to the rectangle contained by LP, OQ as OL to OQ, that is, as OA to ON, and the ſquare of LM is equal to the rectangle OLP, the ſquare of LM will be to the rectangle contained by LP, OQ as OA to ON; therefore the rectangle contained by LP, OQ will be the ſpace to which the ſquare of LM has the ſame ratio that OA has to ON. And therefore the ſquare of EN together with the ſpace to which the ſquare of EO has the ſame ratio that OA has to AN, is equal to the ſquare of NL together with the rectangle OLQ together with the rectangle contained by LP, OQ; but becauſe the rectangle contained by LP, OQ is equal to the rectangle OLP together with the rectangle contained by LP, LQ; therefore the rectangle OLQ together with the rectangle contained by LP, OQ is equal to the rectangle OLP together with the rectangle contained by OP, LQ, that is, equal to the ſquare of LM together with the rectangle contained by OP, LQ; therefore the ſquare of NE together with the ſpace to which the ſquare of EO has the ſame ratio that OA has to AN, is equal to the ſum of the ſquares of LM, LN together with the rectangle contained by OP, LQ. The ſame way it is ſhewn, that the ſquare of FN together with the ſpace to which the ſquare of FO has the ſame ratio that OA has to AN, is equal to the ſum of the ſquares of LM, LN together with the rectangle contained by OP, LQ; and likewiſe, that the ſquare of GN together with the ſpace to which the ſquare of GO has the ſame ratio that OA has to AN, is equal to the ſum of the ſquares of LM, LN together with the rectangle contained by OP, LQ; and ſo on. Therefore the ſum of the ſquares of EN, FN, GN, &c. together

together with the ſpace to which the ſum of the ſquares of EO, FO, GO, &c. has the ſame ratio that OA has to AN, is equal to the multiple of the ſum of the ſquares of LM, LN by the number of of the ſides of the figure together with the multiple of the rectangle contained by OP, LQ by the ſame number.

Again, Becauſe the ſum of the ſquares of EO, FO, GO, &c. is (by the firſt part of this) equal to the multiple of the ſum of the ſquares of LM, LO by number of the ſides of the figure, and the ſquare of LM is equal to the rectangle OLP; the ſum of the ſquares of EO, FO, GO, &c. will be equal to the multiple of the rectangle LOP by the number of the ſides of the figure. And becauſe OA is to AN, as OL to LQ, that is, as the rectangle LOP to the rectangle contained by OP, LQ, that is, as the multiple of the rectangle LOP by the number of the ſides of the figure to the multiple of the rectangle contained by OP, LQ by the ſame number, and the ſum of the ſquares of EO, FO, GO, &c. is equal to the multiple of the rectangle LOP by the number of the ſides of the figure; the ſum of the ſquares of EO, FO, GO, &c. will be to the multiple of the rectangle contained by OP, LQ, as OA to AN; and therefore the multiple of the rectangle contained by OP, LQ by the number of the ſides of the figure, will be the ſpace to which the ſum of the ſquares of EO, FO, GO, &c. has the ſame ratio that OA has to AN; therefore the ſum of the ſquares of EN, FN, GN, &c. together with the multiple of the rectangle contained by OP, LQ by the number of the ſides of the figure is equal to the multiple of the ſum of the ſquares of LM, LN by the number of the ſides of the figure, together with the multiple of the rectangle contained by OP, LQ by the ſame number; and therefore the

ſum

ſum of the ſquares of EN, FN, GN, &c. will be equal to the multiple of the ſum of the ſquares of LM, LN by the number of the ſides of the figure.

PROPOSITION VIII.

THEOREM V. FIG. 18.

Let there be any circle whoſe center is A, and let BCD be a ſemicircle, and BD the diameter of the circle; about the ſemicircle let there be any regular figure deſcribed, and let the ſides of the figure next to BD meet BD in E, F; biſect the ſemicircle in G, and join AG; and in AG take the point H on the ſame ſide the center A with the point G, and let AG be to AH, as the ſum of the ſides of the figure to EF; and let the rectangle HAK be equal to the ſquare of the ſemidiameter, and let HL be equal to AH. If from any point M there be drawn MN, MO, MP, &c. perpendicular to the ſides of the figure circumſcribed about the ſemicircle, and likewiſe let there be drawn ML to the point L; twice the ſum of the ſquares of the perpendiculars MN, MO, MP, &c. will be equal to the multiple of the ſquare of ML by the number of the ſides of the figure together with the multiple of the rectangle KLA by the ſame number.

Let the figure touch the ſemicircle in the points Q, R, S, &c. and join AQ, AR, AS, &c. draw MT, MV, MX, &c. perpendicular to AQ, AR, AS; join MA, MH, and draw HY perpendicular to AH meeting the circle in Y, and join MQ, MR, MS, &c.; becauſe (Prop. VII.) the ſum of the ſquares of MQ, MR, MS, &c is equal to the multiple of the ſum of the ſquares of HM, HY by the number of the ſides of the figure circumſcribed about the ſemicircle,

circle, twice the ſum of the ſquares of MQ, MR, MS, &c. will be equal to twice the multiple of the ſum of the ſquares of HM, HY by the number of the ſides of the figure; therefore twice the ſum of the ſquares of MQ, MR, MS, &c. together with twice the multiple of the ſquare of AH by the number of the ſides of the figure is equal to twice the multiple of the ſum of the ſquares of HM, HA by the number of the ſides of the figure together with twice the multiple of the ſquare HY by the ſame number. And becauſe twice the ſum of the ſquares of HM, HA is equal to the ſum of the ſquares of ML, MA, twice the multiple of the ſum of the ſquares of HM, HA by the number of the ſides of the figure, will be equal to the multiple of the ſum of the ſquares of ML, MA by the ſame number; therefore twice the ſum of the ſquares of MQ, MR, MS, &c. together with twice the multiple of the ſquare of AH by the number of the ſides of the figure, is equal to the multiple of the ſum of the ſquares of ML, MA by the ſame number. But becauſe twice the ſum of the ſquares of MN, MO, MP, &c. together with twice the ſum of the ſquares of MT, MV, MX, &c. is equal to twice the ſum of the ſquares of MQ, MR, MS, &c. therefore twice the ſum of the ſquares of MN, MO, MP, &c. together with twice the ſum of the ſquares of MT, MV, MX, &c. is equal to the multiple of the ſum of the ſquares of ML, MA by the number of the ſides of the figure, together with twice the multiple of the ſquare of HY by the ſame number.

Again, Becauſe the angles MTA, MVA, MXA are right, the points T, V, X will be in the cirumference of the circle whoſe diameter is AM; and becauſe AQ, AR, AS, &c. make all the angles about the point A equal, the circumference of this circle will be

be divided into equal parts in the points T, V, X. &c. (*Lem.* II.) as many in number as there are lines AQ, AR, AS, &c. that is, into as many equal parts as there are ſides in the circumſcribed figure; therefore twice the ſum of the ſquares of MT, MV, MX, &c. will be equal to the multiple of the ſquare of MA by the number of the ſides of the figure; therefore twice the ſum of the ſquares of MN, MO, MP, &c. together with the multiple of the ſquare of MA by the number of the ſides of the figure together with twice the multiple of the ſquare of AH by the ſame number, is equal to the multiple of the ſum of the ſquares of ML, MA by the number of the ſides of the figure together with twice the multiple of the ſquare of HY by the ſame number; and therefore twice the ſum of the ſquares of MN, MO, MP, &c. together with twice the multiple of the ſquare of AH by the number of the ſides of the figure, is equal to the multiple of the ſquare of ML by the number of the ſides of the figure together with twice the multiple of the ſquare of HY by the ſame number.

Again, Becauſe the rectangle HAK is equal to the ſquare of the ſemidiameter of the circle, that is, equal to the ſum of the ſquares of AH, HY; the rectangle KHA, that is, the rectangle KHL, will be equal to the ſquare of HY; and therefore twice the multiple of the rectangle KHL by the number of the ſides of the figure, will be equal to twice the multiple of the ſquare of HY by the ſame number; therefore twice the ſum of the ſquares of MN, MO, MP, &c. together with twice the multiple of the ſquare of AH, or HL by the number of the ſides of the figure, is equal to the multiple of the ſquare of ML by the number of the ſides of the figure together with twice the multiple of the rectangle KHL by the ſame number; therefore twice the ſum of the ſquares of

MN, MO, MP, &c. is equal to the multiple of the ſquare of ML by the number of the ſides of the figure together with the multiple of the rectangle KLA by the ſame number. *Q. E. D.*

Cor. Let there be any equilateral figure inſcribed in a ſemicircle; a point is given ſuch, that if from any point there be drawn perpendiculars to the ſides of the figure, and likewiſe a right line to the given point, twice the ſum of the ſquares of the perpendiculars will be equal to the multiple of the ſquare of the line drawn to the given point by the number of the ſides of the figure together with a given ſpace.

Let ABCD (fig. 19.) be an equilateral figure inſcribed in a ſemicircle; let AD be the diameter, and F the centre; biſect the ſemicircle in G, and join FG; let FH be perpendicular to AB one of the ſides of the figure; in FG take the point K, and let FH be to FK as the ſum of the ſides of the figure ABCD to AD; let KL be equal to FK, and let the rectangle KFM be equal to the ſquare of FH. If from any point N there be drawn NO, NP, NQ, &c. perpendicular to AB, BC, CD, &c. the ſides of the figure, and likewiſe there be drawn NL to the point L, twice the ſum of the ſquares of NO, NP, NQ, &c. will be equal to the multiple of the ſquare of NL by the number of the ſides of the figure together with the multiple of the rectangle MLF by the ſame number.

ARTICLE III.

LUCUBRATIONS IN SPHERICS.

By Mr. JOHN LOWRY,

Officer in his Majeſty's Revenue of Exciſe, at SELI-HULL, near BIRMINGHAM.

PROP. I. THEO. Fig. 20.

IN any right angled ſperical triangle, the rectangle contained by the ſines of the two ſides about the right angle, is equal to the rectangle contained by the ſines of the hypothenuſe and perpendicular falling thereon from the right angle, that is, S. AB × S. BC = S. AC. S. BD.

Demon. For R : S.AB : : S.∠A : S.BD } By Prop.
and R : S. AC : : S.∠A : S.BC } XVIII. Sim. Eu. p. 503.
hence S.AB : SAC : : S.BD : S.BC
and S.AB × S.BC = S.AC × S.BD.

Q. E. D.

PROP. II. THEO. Fig. 21.

In any ſpherical triangle it will be as radius is to the ſine of the vertical angle, ſo is the rectangle contained

ARTICLE III.

LUCUBRATIONS IN SPHERICS.

By Mr. JOHN LOWRY,
Officer in his Majesty's Revenue of Excise, at SELIHULL, near BIRMINGHAM.

PROP. I. THEO. Fig. 20.

IN any right angled sperical triangle, the rectangle contained by the sines of the two sides about the right angle, is equal to the rectangle contained by the sines of the hypothenuse and perpendicular falling thereon from the right angle, that is, S. AB × S. BC = S. AC. S. BD.

Demon. F... : S.AB : : S.∠A : S.BD
: S. AC : : S.∠A : S.BC
AB : SAC : : S.BD : S.BC
.AB × S.BC = S.AC × S.BD.

Q. E. D.

PRO... THEO. Fig. 21.

...ngle it will be as radius is to ...al angle, so is the rectangle con...

tained by the ſines of the ſides to the rectangle contained by the ſines of the baſe and perpendicular.

Demon. Let the arch BQ be drawn to make the angle AQB = ABC,

S. ABC : S. BAC : : S. AC : S. BC, by Trig.
and S. AQB (ABC) : S. BAC : : S. AB : S. QB;
hence S. AC : S. AB : : S. BC : S. QB by equality
or S. AC × S. QB = S. AB × S. BC.

Again, ſince S. DQB = Sine of its ſup. AQB = S. ABC,
it will be R : S. QB : : S. ABC (DQB) : S. DB,
or S. QB = S. DB × (R ÷ S. ABC);
therefore by ſubſtituting for S. QB its equal,
S. AC × S. BD × (R ÷ S. ABC) = S. AB × S. BC,
or R : S. ABC : : S. AB × S. BC : S. AC × S. DB.

Q. E. D.

The ſame demonſtrated in another way.

Firſt. S. AC : S. ABC : : S. BC : S. BAC, by Trig.
therefore S. AC × S. BAC = S. BC × S. ABC,
or (S. AC × S. ABC × R) ÷ R = S. BC × S. ABC;
hence S. AC × S. ABC) ÷ R : S. BC : : S. ABC : R,
and by mult. by S. AB, we have
S. AC × (S. AB × S. BAC) ÷ R : S. BC × S. AB : : S. ABC : R,

Again, R : S. AB : : S. BAC : S. DB,
or S. DB = (SAB × S. BAC) ÷ R,
hence by ſubſtituting for S. DB, we have
R : S. ABC : : S. BC × S. AB : S. AC × S. DB.

Q. E. D.

Cor. If Q be put to denote half the ſum of the three ſides, then S. AC × S. DB will be equal to

$2\sqrt{S.Q \times S.(Q-AC) \times S.(Q-AB) \times S.(Q-BC)}$.

For

For by Art. 168. Crakelt's Trig. $S.\ ABC = 2\ R \times \frac{\sqrt{S.Q \times S.(Q-AB) \times S.(Q-AC) \times S.(Q-BC)}}{S.\ AB \times S.\ BC}$,

or $S.\ AB \times S.\ BC \times S.\ ABC$ is equal to

$2R \times \sqrt{S.Q \times S.(Q-AB) \times S.(Q-AC) \times S.(Q-BC)}$

but by this propofition we have

$S.\ AB \times S.BC \times S.ABC = S.AC \times S.DB \times R$;

therefore $S.\ AC \times S.\ DB$ is equal to

$2\sqrt{S.Q \times S.(Q-AB) \times S\ (Q-AC) \times S.(Q-BC)}$.

PROP. III. THEO. Fig. 22.

In any fpherical triangle it will be as twice the radius is to the tangent of half the vertical angle, fo is the rectangle contained by the fines of the bafe and perpendicular, to the rectangle contained by the fines of the fegments of the bafe made by the point of contact of the infcribed circle.

Demon Put $S =$ fine of half the vertical angle ABC, and $T =$ its tan. and bifect the bafe AC in Q, Then fince $FB = BE$, $AF = AD$, and $CE = CD$,

$AB - BC$ will be $= AD - DC = 2\,QD$;

hence by the laft propofition of Simfon's Euc.

$S.\ AB \times S.\ BC : S.\ AD \times S.\ DC :: R^2 : S^2$,

and by Propofition II.

$S.\ AB \times S.\ BC : S.\ AC \times S.\ BP :: R : S.\ ABC$,

but by Scho. 1. Prop. II. Em. Trig. $S.ABC = 2S^2 \div T$,

therefore $S.AB \times S.BC : S.AC \times S.\ BP :: R : 2S^2 \div T$;

$:: R^2 : 2RS^2 \div T$;

hence $S\ AD \times S.DC : S.AC \times S.BP :: S^2 : 2RS^2 \div T$,

or $S.\ AD \times S.\ DC \times 2R = S.\ AC \times S.\ BP \times T$;

wherefore $2R : T :: S.\ AC \times S.\ BP : S.\ AD \times S.\ DC$.

Q. E. D.

Cor. When the triangle is right angled at B, then T = R, and S.AC × S.BP (= S.AB × S.BC, by Prop. I.) : S.AD × S.DC : : 2 : 1.

PROP. IV. THEO. Fig. 23.

In any ſpherical triangle it will be as twice the radius is to the cotangent of half the vertical angle, ſo is the rectangle contained under the ſines of the baſe and perpendicular, to the rectangle contained by the ſines of half the perimeter, and half the difference between the ſum of the ſides and baſe.

Demon. Let the circles be deſcribed as in the fig. then it is well known that OCQ = half the vertical angle, CQ = half the difference between the ſum of the ſides and baſe, and CT = half the perimeter of the triangle ACB.

Whence by Prop. 42. Cor. 4. Book 3. Em. Trig.

$S.AC \times S.BC : R^2 :: S.TC \times S.QC : \text{coſin.}^2\, OCQ$;

and by Propoſition II. we have

$S.AC \times S.BC : R :: S.AB \times S.CP : S.ACB$,

or $S.AC \times S.BC : R^2 :: S.AB \times S.CP : S.ACB \times R$;

therefore by equality it will be

$S.TC \times S.QC : S.AB \times S.CP :: \text{coſ.}^2\, OCQ : S.ACB \times R$

but $S.ACB = 2\,\text{coſ.}^2\, OCQ \div \text{cot.}\, OCQ$, and therefore

$S.TC \times S.QC : S.AB \times S.CP :: \text{coſ.}^2\, OCQ : (2\,\text{coſ.}^2\, OCQ \div \text{cot.}\, OCQ) \times R$,

or $S.TC \times S.QC \times 2R = S.AB \times S.CP \times \text{cot.}\, OCQ$,

that is, $2R : \text{cot.}\, OCQ :: S.AB \times S.CP : S.TC \times S.QC$.

Q. E. D.

Cor. When the triangle is right angled at C, then cotan. = R, and S. AB × S. CP : S.TC × S. QC : : 2 : 1

PROP.

PROP. V. THEO. Fig. 24.

If from the three angular points of any ſperical triangle, perpendiculars be demitted upon the oppoſite ſides; I ſay, the rectangles contained by the ſine of each perpendicular, and the ſine of the ſide it falls upon are equal.

Demon. By Propoſition II. we have
S. AC × S. CB : S. AB × S. CQ : : R : S. ACB;
and S. AC × S. AB : S. CB × S. AP : : R : S. BAC;
alſo by Trig. S. ACB : S. AB : : S. BAC : S. BC,
or S. ACB : S. AB × S. AC : : S. BAC : S. BC × S. AC;
hence S. AC × S. BC × S. ACB = S. AB × S. AC × S. BAC,
that is S. AB × S. CQ × R = S. BC × S. AP × R,
or S. AB × S. CQ = S. BC × S. AP = (by the ſame method of reaſoning) S. AC × S. BR.

Q. E. D.

Cor. S. CQ × S. ACB = S. AP × S. BAC = S. BR × S. ABC.

For S. AB : S. BC : : S. AP : S. CQ,
and S. AB : S. BC : : S. ACB : S. BAC;
therefore S. AP : S. ACB : : S. CQ : S. BAC;
hence S. CQ × S. ACB = S. AP × S. BAC = (in the ſame way) S. BR × S. ABC.

PROP. VI. THEO. Fig. 21.

In any ſpherical triangle it will be as radius is to the ſine of the baſe, ſo is the rectangle of the ſines of the angles at the baſe to the rectangle of the ſines of the perpendicular and vertical angle.

Demon.

Demon. First S. AC : S. BC : : S. ABC : S. CAB,
and S. BC : R : : S. DB : S. ACB ;
hence by compounding, &c.
S. AC : R : : S. ABC $\times$ S. DB : S. CAB $\times$ S. ACB,
or R : S. AC : : S. CAB $\times$ S. ACB : S. ABC $\times$ S. DB.

Q. E. D.

Cor. I say S. AB $\times$ S. BC : S. $\angle$A $\times$ S. $\angle$C : : sin.[2] AC : sin.[2] ABC.

For by this Prop. and Pro. II. we have
S. $\angle$ A $\times$ S. $\angle$C : S. ABC $\times$ S. BD : : R : S. AC,
and S.BD=(S.AB $\times$ S.BC $\times$ S.ABC) $\div$ (S.AC $\times$ R);
therefore by substituting for S. DB and mult.
S. $\angle$A $\times$ S. $\angle$C $\times$ sin.[2] AC = S.AB $\times$ S.BC $\times$ sin.[2] ABC
or S.AB $\times$ S.BC : S. $\angle$A $\times$ S. $\angle$C : : sin.[2] AC : sin.[2] ABC.

PROP. VII. THEO. Fig. 25.

If about the three angular points of any spherical triangle A, B, C as poles, great circles be described intersecting in D, E and F, and through F and B a great circle be described to cut the bases AC, DE in P and Q; I say, FQ is the supplement of BP.

Demon. Produce the sides as in the figure;
then since CK = CF = a quadrant,
and AM = AF = a quadrant,
F is the pole of AC, and B is the pole of DE;
therefore FP = BQ = a quadrant,
and FQ = QB + PF — PB = a semicircle — PB.

Q. E. D.

Cor. 1. Since FBPQ passes through the poles of AC and DE, the angles at P and Q will be right angles.

Cor.

Cor. 2. $DQ + \angle PBC = QE + \angle PBA =$ a quadrant. For QR is the meaſure of the $\angle$ PBC, and IQ that of PBA; hence $DQ + QR = 90° = QE + QI$.

Cor. 3. $FB = PQ$, $PB = QL = MB = AI$, and $KB = CR$.

Cor. 4. Lay off $Db = bE$, then bQ is the meaſure of half the difference of the angles at the vertex, that is, the meaſure of half the difference of the angles PBA, PBC.

For $Ib = bR =$ the meaſure of the angle $ABy = CBy$; and by *Cor.* 2. $PBA - PBC = DQ - QE = IQ - QR = 2bQ$, or $bQ = \angle PBy$.

Cor. 5. Lay off $Fd = FD$, and join Ad, AE; then $EAd =$ the difference of the angles (A and C) at the baſe of the triangle ABC.

For EF is the ſup. of the $\angle$ A and DF of $\angle$ C; hence $EF - DF = Ed = \angle C - \angle A =$ the meaſure of the $\angle EAd$.

PROP. VIII. PROBLEM.

Given the baſe, the perpendicular, and the difference of the ſides of a ſpherical triangle to determine it.

If in fig. 22. AC be the given baſe, BP the perpendicular, and QD half the difference of the ſides, we have, by Prop. III. $S.AD \times S.DC : S.AC \times S.BP :: 2R : T$, where three of the terms being given, the fourth is alſo given, and the vertical angle is given.

Hence there is given the baſe, perpendicular, and vertical angle, to deſcribe the triangle, which is elegantly done at queſt. 680., *Gent. Diary*.

If the ſum of the ſides, inſtead of the difference, had been given, the vertical angle might have been determined

determined by Prop. IV. and the problem reduced to the 651st quest. *Gent. Diary.*

If the vertical angle, perpendicular, and difference of the angles at the base had been given, then in the supplemental triangle, there would have been given, by Prop. VII. the base, the difference of the sides, and the perpendicular, the very same as the preceding.

PROP. IX. PROBLEM.

Given the base and perpendicular of a spherical triangle to construct it, when the rectangle contained by the sines of the angles at the base is a maximum.

Conſ. With the given base AC (fig. 20), and perpendicular DB, construct the right-angled triangle ABC, and it will be that required.

Demon. By Prop. VI. as S.AC : R :: S.ABC × S.BD : S.CAB × S.ACB, where the two first terms being constant, it is manifest the fourth will be a max. when the third is so, which will be when ABC is a right angle. *Q. E. D.*

ARTICLE

ARTICLE IV.

NEW TABLES,

FOR FINDING THE CONTENTS OF CASKS.

By Mr. JOHN LOWRY.

Quotient of the Head divided by the Bung.	First Variety, or Middle Frustum of a Spheriod.		Second Variety, or Middle Frust. of a Para. Spindle.	
	A. G'.	W. G'.	A. G'.	W. G'.
·50	·0020888	·0025500	·0019959	·0024366
·51	·0020982	·0025614	·0020089	·0024525
·52	·0021077	·0025731	·0020222	·0024687
·53	·0021175	·0025850	·0020355	·0024849
·54	·0021274	·0025971	·0020489	·0025012
·55	·0021375	·0026094	·0020623	·0025176
·56	·0021478	·0026220	·0020750	·0025332
·57	·0021583	·0026348	·0020897	·0025511
·58	·0021690	·0026479	·0021035	·0025679
·59	·0021798	·0026611	·0021175	·0025850
·60	·0021909	·0026746	·0021315	·0026021
·61	·0022021	·0026883	·0021457	·0026194
·62	·0022136	·0027023	·0021602	·0026371
·63	·0022252	·0027164	·0021744	·0026544
·64	·0022370	·0027308	·0021889	·0026721
·65	·0022489	·0027454	·0022034	·0026899
·66	·0022611	·0027603	·0022182	·0027079

Quotient of the Head divided by the Bung.	First Variety, or Middle Fruſtum of a Spheriod.		Second Variety, or Middle Fruſt. of a Para. Spindle.	
	A. Gs.	W. Gs.	A. Gs.	W. Gs.
·67	·0022734	·0027754	·0022331	·0027261
·68	·0022860	·0027907	·0022480	·0027443
·69	·0022987	·0028062	·0022630	·0027627
·70	·0023116	·0028219	·0022782	·0027811
·71	·0023247	·0028379	·0022934	·0027998
·72	·0023380	·0028541	·0023087	·0028184
·73	·0023514	·0028706	·0023244	·0028376
·74	·0023651	·0028872	·0023437	·0028612
·75	·0023789	·0029040	·0023557	·0028758
·76	·0023929	·0029212	·0023716	·0028952
·77	·0024071	·0029386	·0023875	·0029146
·78	·0024215	·0029561	·0024036	·0029343
·79	·0024361	0029739	·0024197	·0029539
·80	·0024508	·0029919	·0024360	·0029738
·81	·0024658	·0030102	0024524	·0029939
·82	·0024809	·0030287	·0024690	·0030130
·83	·0024962	·0030474	·0024856	·0030343
·84	·0025117	·0030552	·0025023	·0030547
·85	·0025274	·0030854	·0025191	·0030752
·86	·0025433	·0031047	·0025361	·0030960
·87	·0025594	·0031244	·0025532	·0031168
·88	·0025756	·0031443	·0025703	·0031378
·89	·0025921	·0031643	·0025876	·0031589
·90	·0026087	·0031846	·0026050	·0031801
·91	·0026255	·0032052	·0026225	·0032015
·92	·0026425	·0032259	·0026401	·0032230
·93	·0026596	·0032468	·0026579	·0032447
·94	·0026770	·0032680	·0026757	·0032664
·95	·0026945	·0032894	·0026936	·0032883

Quotient of the Head divided by the Bung.	First Variety, or Middle Fruſtum of a Spheriod.		Second Variety, or Middle Fruſt. of a Para. Spindle.	
	A. G^s.	W. G^s.	A. G^s.	W. G^s.
·96	·0027123	·0033111	·0027117	·0033104
·97	·0027302	0033330	·0027299	·0033326
·98	·0027483	·0033551	·0027482	·0033549
·99	·0027666	·0033774	·0027665	·0033773
1·00	·0027851	·0033999	·0027851	·0033999

GENERAL RULE.

Divide the head diameter by the bung diameter to two places of decimals in the quotient, againſt which, in the column anſwering to the propoſed variety, we have two decimals one for *ale*, the other for *wine* gallons; which being multiplied continually by the ſquare of the bung diameter, and the length of the caſk, the laſt product will be the content.

Remark. Theſe Tables are founded on nearly the ſame principles as Mr. *Moſs*'s, given at page 190 of his Treatiſe on Gauging; they are equally accurate, and much readier in practice. The inveſtigation will be given in ſome future number, and the tables extended to the other varieties.

ARTICLE V.

TABLES OF THEOREMS

FOR THE

CALCULATION OF FLUENTS,

From Mr. LANDEN's Memoirs,

Communicated by Mr. WILLIAM BURDON.

TABLE I.

THEOREM I.

$$\dot{F} = x^n \dot{x}.$$

$$F = \frac{x^{n+1} - c^{n+1}}{n+1}.$$

c the quantity to which x is equal when $F = o$.

Note. When n is $= -1$, the expreſſion for the value of

$$F \text{ becomes} = \text{Log. } \frac{x}{c}$$

THEOREM

THEOREM II.

$$\dot{F} = (a^n + b x^n)^p x^{n-1} \dot{x}.$$

$$F = \frac{(a^n + b x^n)^{p+1} - (a^n + b c^n)^{p+1}}{b n . (p+1)}.$$

c the value of x when $F = o$.

Note. When p is $= -1$, the expression for the value of

$$F \text{ becomes } = \frac{1}{bn} \times \text{Log.} \frac{a^n + b x^n}{a^n + b c^n}.$$

THEOREM III.

$$\dot{F} = \frac{x^{np-1} \dot{x}}{(a^n + b x^n)^{p+1}}.$$

$$F = \frac{1}{n p a^n} \left(\frac{x^{np}}{(a^n + b x^n)^p} - \frac{c^{np}}{(a^n + b c^n)^p} \right).$$

c the value of x when $F = o$.

Note. When p is $= o$, the expression for the value of

$$F \text{ becomes } = \frac{1}{n a^n} \times \text{Log.} \frac{x^n}{c^n} \times \frac{a^n + b c^n}{a^n + b x^n}.$$

THEOREM IV.

$$\dot{F} = \frac{x^{n-1} \dot{x}}{\sqrt{2 a x^n + x^{2n}}}.$$

THEOREM XI.

$$\dot{F} = \frac{x^{n-1}\,\dot{x}}{\sqrt{x^{2n} - a^{2n}}}.$$

$$F = K + \frac{1}{na^{2n}} \times \text{Circ. Arc, rad. } a^{n}\text{, fecant. } x^{n}.$$

TABLE

TABLE II.

CONTAINING

THEOREMS

FOR THE

CALCULATION OF FLUENTS.

THEOREM I.

$$\dot{F} = \frac{x^{m}\,\dot{x}}{x+a}.$$

m any positive integer.

$$F = K + a^{m} \times \overline{\frac{x^{m}}{m\,a^{m}} - \frac{x^{m-1}}{\overline{m-1}\,.\,a^{m-1}} + \frac{x^{m-2}}{\overline{m-2}\,.\,a^{m-2}}}$$

$$\overline{(m)\ {}^{*} \pm \text{Log.}\ \overline{x+a}}.$$

* $+$ or $-$ according as m is even or odd.

THEOREM II.

$$\dot{F} = \frac{x^{-m}\,\dot{x}}{x+a}.$$

m any positive integer.

F =

$$F = K + \frac{1}{a^m} \times \overline{\frac{a^{m-1}}{m-1 \cdot x^{m-1}} - \frac{a^{m-2}}{m-2 \cdot x^{m-2}} + \frac{a^{m-3}}{m-3 \cdot x^{m-3}} \cdot}$$

$$\overline{(m-1)} \; {*} \; \pm \text{ Log. } \frac{x+a}{x}.$$

$*$ $+$ or $-$ according as $m-1$ is even or odd.

THEOREM III.

$$\dot{F} = \frac{x^m \dot{x}}{x^2 + a^2}.$$

m any even positive number.

$$F = K + a^{m-2} \times \overline{\frac{x^{m-1}}{m-1 \cdot a^{m-2}} - \frac{x^{m-3}}{m-3 \cdot a^{m-4}} + \frac{x^{m-5}}{m-5 \cdot a^{m-6}} \cdot}$$

$$\overline{\left(\frac{m}{2}\right)} \; {*} \; \pm A.$$

$A =$ Circ. Arc, rad. a, tang. x.

$*$ $+$ or $-$ according as $\frac{m}{2}$ is even or odd.

THEOREM IV.

$$\dot{F} = \frac{x^m \dot{x}}{x^2 - a^2}.$$

m any even positive number.

F

$$\dot{F} = K + a^{m-1} \times \overline{\frac{x^{m-1}}{m-1 \cdot a^{m-1}} + \frac{x^{m-3}}{m-3 \cdot a^{m-3}} + \frac{x^{m-5}}{m-5 \cdot a^{m-5}}} \\ \overline{\left(\frac{m}{2}\right)} + \frac{1}{2} \text{Log.} \frac{a-x}{a+x} \cdot$$

THEOREM V.

$$\dot{F} = \frac{x^{m} \dot{x}}{x^2 + a^2} \cdot$$

m any odd positive number.

$$F = K + a^{m-1} \times \overline{\frac{x^{m-1}}{m-1 \cdot a^{m-1}} - \frac{x^{m-3}}{m-3 \cdot a^{m-3}} + \frac{x^{m-5}}{m-5 \cdot a^{m-5}}} \\ \overline{\left(\frac{m-1}{2}\right)} \;{}^{*} \pm \tfrac{1}{2} \text{Log.} \left(x^2 + a^2\right) \cdot$$

a^2 either positive or negative.

* $+$ or $-$ according as $\frac{m-1}{2}$ is even or odd.

THEOREM VI.

$$\dot{F} = \frac{x^{-m} \dot{x}}{x^2 + a^2} \cdot$$

m any even positive number.

F =

$$F = K + \frac{1}{a^{m+1}} \times \overline{\frac{a^{m-1}}{m-1 \,.\, x^{m-1}} - \frac{a^{m-3}}{m-3 \,.\, x^{m-3}} + \frac{a^{m-5}}{m-5 \,.\, x^{m-5}} \ldots \left(\frac{m}{2}\right)} \; {}^{*} \pm A.$$

$A =$ Circ. Arc, rad. a, tang. $\frac{a}{x}$

* $+$ or $-$ according as $\frac{m}{2}$ is even or odd.

THEOREM VII.

$$\dot{F} = \frac{x^{-m} \, \dot{x}}{x^2 - a^2}.$$

m any even positive number.

$$F = K + \frac{1}{a^{m+1}} \times \overline{\frac{a^{m-1}}{m-1 \,.\, x^{m-1}} + \frac{a^{m-3}}{m-3 \,.\, x^{m-3}} + \frac{a^{m-5}}{m-5 \,.\, x^{m-5}} \ldots \left(\frac{m}{2}\right)} + \tfrac{1}{2} \text{ Log. } \frac{x-a}{x+a}.$$

THEOREM VIII.

$$\dot{F} = \frac{x^{-m} \, \dot{x}}{x^2 + a^2}.$$

m any odd positive number.

$F =$

$$F = K + \frac{1}{a^{m+1}} \times \overline{\frac{a^{m-1}}{m-1 \,.\, x^{m-1}} - \frac{a^{m-3}}{m-3 \,.\, x^{m-3}} + \frac{a^{m-5}}{m-5 \,.\, x^{m-5}}}$$

$$\overline{\left(\frac{m-1}{2}\right)^{*} \pm \text{Log.}\, \frac{x^2 + a^2}{x^2}}$$

a^2 either positive or negative.

* $+$ or $-$ according as $\frac{m-1}{2}$ is even or odd.

TABLE

TABLE III.

CONTAINING THEOREMS FOR THE CALCULATION OF FLUENTS.

THEOREM I.

$$\dot{F} = \frac{x^{-\frac{1}{2}}\,\dot{x}}{\sqrt{a^2 - x^2}}.$$

$$F = K + \frac{4}{a^{\frac{3}{2}}} \times \overline{de - \grave{e}'e''} = K + \frac{2}{a^{\frac{3}{2}}} \times \overline{de + DP - AD - L}.$$

THEOREM II.

The fluent of $\frac{x^{-\frac{1}{2}}\,\dot{x}}{\sqrt{a^2 - x^2}}$, generated whilst x from o becomes equal to any quantity k, is equal to the fluent of the same fluxion, generated whilst x from $a \times \frac{a - k}{a + k}$ becomes equal to a.

Note.

Note. All the Theorems in this table refer to the Scheme at the end of it, for the values of the quantities required.

THEOREM III.

The fluent of $\frac{x^{-\frac{1}{2}}\dot{x}}{\sqrt{a^2-x^2}}$, generated whilst x from becomes equal to $\overline{2^{\frac{1}{2}}-1} \times a$, is $= \frac{M}{a^{\frac{3}{2}}}$.

THEOREM IV.

The *whole* fluent of $\frac{x^{-\frac{1}{2}}\dot{x}}{\sqrt{a^2-x^2}}$ is $= \frac{2\,M}{a^{\frac{3}{2}}}$

THEOREM V.

$$\dot{F} = \frac{x^{\frac{1}{2}}\dot{x}}{\sqrt{a^2-x^2}}$$

$$F = K + \frac{2}{a^{\frac{1}{2}}} \times \overline{2\,de'' - de} = K + \frac{2}{a^{\frac{1}{2}}} \times \overline{L + AD - DP}.$$

THEOREM VI.

The tangent co $\left(-\overline{ax}|^{\frac{1}{2}} \times \overline{\frac{a-x}{a+x}}|^{\frac{1}{2}}\right)$ together with the fluent of $\frac{\frac{1}{2} a^{\frac{1}{2}} x^{\frac{1}{2}} \dot{x}}{\sqrt{a^2-x^2}}$, generated whilſt x from o becomes equal to any quantity k, is equal to the fluent of the ſame fluxion, generated whilſt x from $a \times \frac{a-k}{a+k}$ becomes equal to a.

THEOREM VII.

The fluent of $\frac{x^{\frac{1}{2}} \dot{x}}{\sqrt{a^2-x^2}}$, generated whilſt x from o becomes equal to $\overline{2^{\frac{1}{2}}-1} \times a$, is $= \frac{L}{a^{\frac{1}{2}}} - \overline{2^{\frac{1}{2}}-1} \times a^{\frac{1}{2}}$.

THEOREM VIII.

The *whole* fluent of $\frac{x^{\frac{1}{2}} \dot{x}}{\sqrt{a^2-x^2}}$ is $= \frac{2L}{a^{\frac{1}{2}}}$.

THEOREM

THEOREM IX.

$$\dot{F} = \frac{y^{-\frac{1}{2}}\,\dot{y}}{\sqrt{y^2 - a^2}}.$$

$$F = K + \frac{4}{a^{\frac{3}{2}}} \times \overline{ae - E'' + e'e'} = K + \frac{2}{a^{\frac{1}{2}}} \times$$
$$\overline{ae + AD - DP}.$$

$$x = \frac{a^2}{y}.$$

THEOREM X.

The fluent of $\frac{y^{-\frac{1}{2}}\,\dot{y}}{\sqrt{y^2 - a^2}}$ generated whilst y from a becomes equal to $\overline{2^{\frac{1}{2}} + 1} \times a$, is $= \frac{M}{a^{\frac{1}{2}}}$.

THEOREM XI.

The *whole* fluent of $\frac{y^{-\frac{1}{2}}\,\dot{y}}{\sqrt{y^2 - a^2}}$ is $= \frac{2M}{a^{\frac{1}{2}}}$.

THEOREM XII.

$$\dot{F} = \frac{y^{\frac{1}{2}}\dot{y}}{\sqrt{y^2 - a^2}}.$$

$$F = K + \frac{2}{a^{\frac{1}{2}}} \times \overline{DP + ae + 2e'e'' - 2E'''}$$

$$= K + \frac{2}{a^{\frac{1}{2}}} \times AD.$$

$$x = \frac{a^2}{y}.$$

THEOREM XIII.

The fluent of $\frac{y^{\frac{1}{2}}\dot{y}}{\sqrt{y^2 - a^2}}$, generated whilst y from a becomes equal to $\overline{2^{\frac{1}{2}} + 1} \times a$, is $= \overline{2^{\frac{1}{2}} + 1} \times a^{\frac{1}{2}}$ $- \frac{L}{a^{\frac{1}{2}}}$.

Note. The *whole* fluent is infinite.

THEOREM XIV.

$$\dot{F} = \frac{y^{-2}\dot{y}}{\sqrt{a^2 + y^2}}.$$

F =

$$\text{F} = \text{K} + \frac{2^{\frac{1}{2}}}{a^{\frac{3}{2}}} \times \overline{\text{ae} + e'e'' - \text{E}''} = \text{K} + \frac{2^{\frac{3}{2}}}{a^{\frac{3}{2}}} \times \overline{\text{ae} + \text{AD} - \text{DP}}.$$

$$x = \overline{a^2 + y^2}\Big|^{\frac{1}{2}} - y.$$

THEOREM XV.

The fluent of $\frac{y^{-\frac{1}{2}}\dot{y}}{\sqrt{a^2 + y^2}}$, generated whilst y from o becomes equal to a, is $= \frac{2^{\frac{1}{2}}}{a^{\frac{3}{2}}} \times \text{M}$.

THEOREM XVI.

The *whole* fluent of $\frac{y^{-\frac{1}{2}}\dot{y}}{\sqrt{a^2 + y^2}}$ is $= \frac{2^{\frac{3}{2}}}{a^{\frac{3}{2}}} \times \text{M}$.

THEOREM. XVII.

$$\dot{\text{F}} = \frac{y^{\frac{1}{2}}\dot{y}}{\sqrt{a^2 + y^2}}.$$

$$\text{F} = \text{K} + \frac{2^{\frac{1}{2}}}{a^{\frac{1}{2}}} \times \overline{\text{ae} + 2\,e'e'' - 2\,\text{E}'' + \tfrac{1}{2}\,\text{DP}}$$

$$= \text{K} + \frac{2^{\frac{3}{2}}}{a^{\frac{1}{2}}} \times \overline{\text{AD} - \tfrac{1}{2}\,\text{DP}}.$$

$$x = \overline{a^2 + y^2}\Big|^{\frac{1}{2}} - y.$$

 THEOREM

THEOREM XVIII.

The fluent of $\frac{y^{\frac{1}{2}}\dot{y}}{\sqrt{a^2+y^2}}$, generated whilst y from o becomes equal to a, is $= \sqrt{2a - \frac{2}{a}}\Big|^{\frac{1}{2}} \times L$.

Note. The *whole* fluent is infinite.

THEOREM XIX.

$$\dot{F} = \frac{\dot{y}}{\overline{a^2 - y^2}|^{\frac{1}{4}}}.$$

$F = K + \frac{2}{a^{\frac{1}{2}}} \times \overline{2E'' - 2e'e'' - ae} = K + \frac{2}{a^{\frac{1}{2}}} \times \overline{DP - AD}.$

$$x = \sqrt{a^2 - y^2}$$

THEOREM XX.

The fluent of $\frac{\dot{y}}{\overline{a^2-y^2}|^{\frac{1}{4}}}$, generated whilst y from o becomes equal to $\sqrt{2^{\frac{3}{4}} - 2} \times a$, is $= \frac{L}{a^{\frac{1}{2}}} + \overline{2^{\frac{1}{2}} - 1} \times a^{\frac{1}{2}}$.

ARTICLE VI.

A COLLECTION OF PROBLEMS

To be anſwered in Number III.

I. QUESTION 9, *by Juvenis Mathematicus.*

From the data, dear Gents, which are placed below,
The greateſt of *ſecrets* I would have you to ſhew.

$$\left.\begin{array}{l} x^2+z^2-2yz+y^2=170=a \\ y^2+z^2-2zx+x^2=234=m \\ z^2+y^2-2yx+x^2=80=n \end{array}\right\}$$

Where x, y and z repreſent the places in the alphabet compoſing the ſecret?

II. QUESTION 10, *by Mr. T. Bulmer, Sunderland.*

There is a cone, which being ſuſpended by its vertex, the number of vibrations it makes in a minute, its altitude, and the radius of its baſe in inches, are as 11, 10 and 1:—Required how often it vibrates in a minute and its ſolid content?

III. QUESTION 11, *by Mr. J. Surtees, Sunderland.*

It is required to find the preſſure of water with the velocity of 0·00002 feet per ſecond, againſt a flood-gate placed perpendicular to the horizon, whoſe breadth is 18, and depth 12 feet?

IV.

IV. QUESTION 12, *by Mr. J. Rutherford, Weardale.*

On Midſummer-day, in latitude 54° 40′ north, at 10 o'clock in the forenoon, I obſerved the ſun to ſhine into a ſhaft made for the purpoſe of winding up the ore got in the mines; the declination of the ſhaft I found was S. S. W. and breadth 4 feet:—*Query* the depth to which the ſun ſhined therein, and the length of his longeſt ray from the upper edge of the ſhaft, to the loweſt point enlightened thereby on the oppoſite ſide?

V. QUESTION 13, *by Mr. Burdon, Acaſter-Malbis.*

There are three towns A, B and C, the roads to which, from one another, form a right-angled triangle. Now a perſon had to travel from the town B at the right angle to A; but after going two miles, had occaſion to call ſomewhere on the road from A to C; he therefore takes the neareſt way to it, and then finds he is one mile and a half from A and three from C:—*Query* the diſtance from B to C, and the number of miles he had travelled when arrived at A?

VI. QUESTION 14, *by Plus-Minus, Selby.*

A perſon of my acquaintance has an equilateral triangular yard to be divided into three parts by paling, drawn from the center of a baſon, ſomewhere within it, to the neareſt point in each ſide. Now he is informed that it will coſt him as much doing at 12s. 6d. per pole, as the whole yard would paving at 9d. per yard:—*Query* the ſides and area of the ſaid yard?

VII.

VII. QUESTION 15, *by Mr. Collin Campbell, Kendal.*

If FGHI, KCML (fig. 31.) be two wheels, revolving round the centres S, O, and connected by the flexible band FGHMLKF. It is required to determine the friction of that band on each wheel, supposing the center S fixed, and the centre O urged by a force in the direction SO = T.

VIII. QUESTION 16, *by Mr. J. Fletcher, Liverpool.*

Seeing an exciseman's staff in form of a cylinder, three-fourths of an inch in diameter, and thirty-six inches long, immersed in a vessel of beer at one end, the other resting on the edge of the vessel 3 inches above the liquor, I observed 13 inches along the staff's axis to be dry:—Required the weight of the staff? a cubic inch of beer weighing 0.5949 oz. aver.

IX. QUESTION 17, *from Lawson on the Ancient Analysis.*

Let there be a triangle ABC, whose base BC is bisected in D, and through the vertex A a line AE drawn parallel to BC, and any line drawn through D to meet AB, AC, AE in F, G, H; then I say GD : DF : : GH : HF:—Required the demonstration?

X. QUESTION 18, *from the same.*

If in AB the diameter of a circle two points C and D be taken such that AC : CB : : AD : DB, and through the point D any line be drawn to meet the circle in E and F, and CE, CF be joined; then I say

ſay EC : CF : :.ED : DF :—Required the demonſtration ?

XI. QUESTION 19, *from Stewart's General Theorems.*

Let there be any number of given points A, B, C, &c. and let *a*, *b*, *c*, &c. be given magnitudes as many in number as there are given points; a point X may be found, ſuch, that if from the given points A, B, C, &c. there be drawn right lines to the point X, and from the given points and the point X there be drawn right lines to any point Y, the ſquare of AY together with the ſpace to which the ſquare of BY has the ſame ratio that *a* has to *b*, together with the ſpace to which the ſquare of CY has the ſame ratio that *a* has to *c*, and ſo on, will be equal to the ſquare of AX together with the ſpace to which the ſquare of BX has the ſame ratio that *a* has to *b*, together with the ſpace to which the ſquare of CX has the ſame ratio that *a* has to *c*, and ſo on, together with the ſpace to which the ſquare of XY has the ſame ratio that *a*. has to the ſum of *a*, *b*, *c*, &c.—Required the demonſtration ?

XII. QUESTION 20, *by Mr. R. Simpſon.*

Bartered a piece of broad-cloth, containing *a* yards at *b* ſhillings per yard, for a piece of fine Iriſh linen and another of cambric. Now the ratio of the yards in theſe two pieces was that of *c* to *d*, and the ratio of their values per yard, in ſhillings, that of *m* to *n*; alſo the rated price of the linen per yard, was to the number of yards in the piece as *r* to *s* :—Required the yards, prices per yard, and values of the two pieces ?

QUESTION

XIII. QUESTION 21, *by Mr. Olinthus Gilbert Gregory.*

The axis of a ſphere is 12 inches; what is the difference between the ſolidity of this ſphere, and that of a cone whoſe ſlant height is to the radius of its baſe as 3 to 1, and the whole ſurface equal to the ſurface of the ſphere?

XIV. QUESTION 22, *by Mr. John Lowry.*

If tangents be drawn from the extremities of a given oblique parabola; it is required to determine the area of the greateſt ellipſis that can be inſcribed in the ſpace included between the tangents and the curve?

XV. QUESTION 23, *by Mr. Lowry.*

Given the perimeter, the vertical angle and area of a ſpherical triangle to determine it?

XVI. QUESTION 24, *by Mr. W. Pearſon, North Shields.*

The fluent of $\overline{a+cz^n}|^m \times z^{pn-1}\dot{z}$ being given, from p. 94. of *Simpſon's Fluxions,* it is required to find the fluents of $\overline{a+cz^n}|^{m-r} \times z^{pn+vn-1}\dot{z}$, of $\overline{a+cz^n}|^{m+r} \times z^{pn-vn-1}\dot{z}$, and alſo of $\overline{a+cz^n}|^{m-r} \times z^{pn-vn-1}\dot{z}$?

QUESTION

XVII. QUESTION 25, *by Pappus, junior.*

In the ſtraight line AB, take BC ſo that AC may be triple of CB; and let D be any point in AB; I ſay, the ratio of the cube of DA to the cube of AC will be leſs than the ratio of BC to BD. Required the demonſtration?

XVIII. QUESTION 26, *by Mr. Thomas Leybourn.*

Given $\frac{p\dot{x}}{x} + \frac{r\dot{y}}{y} = \frac{x^{m}\dot{x}}{ay^{n}}$; to find the relation of the fluents?

XIX. QUESTION 27, *by Tommy Fluxion.*

Required the fluxion of $\overline{x-y}|^{z^{v}}$ where all the quantities are variable?

XX. QUESTION 28, *being Prop. 2nd. of Mac Laurin's Geometria Organica.*

Determinare Curvarum Aſſymptotos atque Species.

Required an Engliſh tranſlation and ſolution, according to the author's method, as alſo any other method.

THE

MATHEMATICAL REPOSITORY.

ARTICLE VII.

A curious Problem, with its Investigation, from the Gentleman's Magazine, for May, 1768, by Mr. THOMAS TODD, of Scorton, (late of West-Smithfield, London.)

PROBLEM. *Fig.* 33, *Plate* 2.

REQUIRED the value of a Solid, generated by the rotation of any hyperbolic Segment, round an ordinate as axis? And alſo the Segments GeGI, cut off by circular ſections GeG, perpendicular to the ſection II.

INVESTIGATION.

If $c = 3{\cdot}14159265$ &c. $m =$ Cb $=$ bO the ſemi-tranſverſe diameter, $n =$ bB its correſpondent ſemi-conjugate, $r =$ bN $=$ dR, $s =$ nat. ſine of angle GRe or OND and p its coſine, $h =$ ON, $v =$ GR; then by the hyper. $m : n :: \sqrt{2mh + h^2} : \frac{n}{m}\sqrt{2mh + h^2} =$ IN $= a = \frac{n}{m}\sqrt{r^2 - m^2}$ and put $u =$ dG $=$ bM, $x =$ GM $=$ RN; dB $\parallel$ to GG and II; Ge and OD $\perp$ II; then by the hyper. (prop. 7. page 58, and cor. 3, page 77. *De la Hoſpital's* Con.) we have $m^2 : n^2 :: \overline{u+m}$ (CM) $\times \overline{u-m}$ (OM) $: \frac{n^2}{m^2} \times (u^2 - m^2) = x^2 =$ GM^2; whence bM $= u = \frac{m}{n}\sqrt{n^2 + x^2}$; GR ($=$ MN) $= r - \frac{m}{n}\sqrt{n^2 + x^2}$, and Ge (by trigonometry) $= sr - \frac{sm}{n}\sqrt{n^2 + x^2}$;

$$\text{Ge}^2 = s^2 \times \left(r^2 - \frac{2rm}{n}\sqrt{n^2 + x^2} + m^2 + \frac{m^2 x^2}{n^2}\right),$$

 conſequently

consequently the fluxion of the solid GRGONO $=$ $\dot{S}$ ($= c. \overline{Ge}|^2 \times$ flux. of NR) $= cs^2 \times (r^2 \dot{x} - \frac{2rm\dot{x}}{n} \sqrt{n^2 + x^2} + m^2 \dot{x} + \frac{m^2 x^2 \dot{x}}{n^2})$ whose correct fluent $S = cs^2 \times (r^2 x + m^2 x + \frac{m^2 x^3}{3n^2} - \frac{mrx}{n} \sqrt{n^2 + x^2} - mrn \times$ hyp. log. of $\frac{x + \sqrt{n^2 + x^2}}{n})$ the solid itself, where $S = o$ when $x = o$.

OBSERVATIONS.

First. The Cone ONOD — Cone GRGe $+ S =$ the frustum GeGODO on the left side of ODO. Cone GRGe — Cone ONOD $+ S =$ frustum GeGODO on the right side of ODO; and the sum of the frustums $= 2S =$ GOGGOG the two frustums together when the lesser diameters are equal. And when $x = a =$ IN, we get the whole solid $= 2cs^2 \times (r^2 a + m^2 a + \frac{m^2 a^3}{3n^2} - \frac{mra}{n} \sqrt{n^2 + a^2} - mrn \times$ hyp. log. of $\frac{a + \sqrt{n^2 + a^2}}{n} =$ the whole solid generated by the segment IOI round II, and one half of this is half the solid $=$ IONO.

Second. The greater segment GeGI $= \frac{1}{2}$ solid ONOI — solid ONOGRG (S) $+$ Cone GRGe; and the lesser segment GeGI $= \frac{1}{2}$ solid ONOI — solid ONOGRG (S) — Cone GRGe. And, if to $\frac{1}{2}$ the solid ONOI, we add and subtract the Cone ONOD, the sum and difference, will give the two solids IODO, parted by the circular section ODO.

Third. The Cones GRGe, ONOD (generated in the solid) mentioned above are thus found, *viz.* Rad. $1 : v :: p : pv =$ Re, and Rad. $1 : v :: s : sv =$ Ge therefore the Cone GRGe $= \frac{1}{3} cs^2 pv^3$, and in like manner Cone ONOG $= \frac{1}{3} cs^2 ph^3$.

Fourth.

Fourth. There will another problem arife, *viz.* to find a fegment GOG cut off parallel to the fixed axis II.

SCHOLIUM.

The above is an explanation of the Gentleman's Mag. folution, mentioned above, and the fluxion and fluent S of the folid GRGONO, found above, is of the very fame value as that given in the Magazine, and half the folid IONO found from it, when $x = a =$ IN. Then from thefe as *data*, and the nature of the folid, the fruftums and fegments are in terms given or known, as fhewn in the above folution. Your readers may compare this with the remarks and fcholium, made at the end of the folutions to this Queftion, in the firft and fecond editions of a book of Menfuration. In the quarto edition, page 400, the folution is corrected in page 21, 22, of the *Mifcellanea Mathematica.* This folution proves the falfity of the cenfure againft the magazine folution.

In juftice to myfelf, I wifh to inform the readers of the REPOSITORY and others, that two of my folutions in *Britifh Diary* for 1792, one in page 41, 42, and the other in page 43, 44, are both fpoiled by the Editor and Printer of that Diary; and the figures and letters are not as I fent them. Alfo queftion 8, page 47, of the fame Diary, by *Philalathes Cleafbyenfis*, (one would think by defign) is made nonfenfe of. And the true values of y and x to queftion 2, page 33, of the fame Diary, are $y = 1{\cdot}013718$, and $x = 432{\cdot}3741$, numbers very different from thofe given in that Diary. Once more, (with the Editor's permiffion) I invite Mr. *Tho. Keith* to give his proof, of what he has boldly affirmed in page 65, of his *Key* or *Arithmetical Appendix*, againft my equatement folution, given in the *Ladies Diary*, for 1789; and as the

ſolution is more explained in No. 10 of the *Scientific Receptacle*, I expect Mr. *Keith* will do his buſineſs effectually, or, like a man who loves truth, when he finds himſelf in errors, acknowledge them. Alſo in Cor. 2, page 64, in his appendix ſolution for four debts, it is expected Mr. *Keith* will ſhew how he knew from his proceſs that his x equated time, would fall between the ſecond and third debts.

To the Editor of the Mathematical Repoſitory.

SIR,

I Have incloſed the above Problem, with the Inveſtigation, Obſervations, and Scholium; which I hope you will do me the favour to inſert in No. 2, of the *REPOSITORY*, and you will oblige,

Sir, your very humble ſervant,

THOMAS TODD.

Scorton, June 10, 1796.

ARTICLE VIII.

A remarkable new Property of the Cycloid diſcovered, which ſuggeſts a new method of regulating the motion of a Clock.

From Mr. LANDEN's MEMOIRS.

LET $A'B'P'E'Q'$, $A''B''P''E''Q''$ (Fig. 29, Plate 1, in No. 1) be two ſimilar curved ſmall tubes, ſituated exactly alike in a vertical plane; let a ſmall ball be ſuppoſed to be put into each tube; and, both the balls P', P'' being equal, let them be conceived to be connected by a perfectly flexible line, without weight, paſſing from P' up the tube wherein it is put to the top A', and from thence to the top A'' of the other tube, then down that other tube to P'': let that flexible line ($P'A'$ $A''P''$) be equal to $E'B'A'$ $A''B''E''$; and $A'A''$, $B'B''$, $E'E''$, $Q'Q''$, being horizontal lines, let $B'E'$, $E'Q'$, $B''E''$, $E''Q''$, be

be all equal, that, the balls being moved, P″ may be at Q″, E″, or B″, when P′ ſhall be at B′, E′, or Q′ reſpectively. Then the ball P′ being raiſed to B′, and left to deſcend from thence in the tube A′B′E′Q′; and the ball P″, during the deſcent of P′, being drawn up the other tube from Q″, by means of the ſaid connecting line; it is propoſed to find the nature of the curve into which the tubes muſt be bent, that the time of deſcent of the ball P′ (ſo connected with the ball P″) from B′ to Q′, may always be the ſame, let the height B′E′ be what it will.

Put a for the length of the part B′E′ of the tube into which the ball P′ is ſuppoſed to be put; b for the vertical height of B′ above E′; z for the ſpace paſſed over by P′ in the tube in its deſcent; x for the vertical deſcent of P′; y for the vertical aſcent of P″; v for the velocity of each of the balls; t for the time elapſed during the deſcent of P′; and g for $32\frac{1}{6}$ feet, the accelerative force of gravity: then will $g\dot{x}P' \div \dot{z}$ be the motive force by which the velocity v will be accelerated, $g\dot{y}P'' \div \dot{z}$ the motive force by which v will be retarded, and $\frac{1}{2}g.\ (\dot{x}-\dot{y}) \div \dot{z}$ the actual accelerating force of each ball. Now, that P′ may always arrive at E′ in the ſame time, let the diſtance B′E′ be what it will, its accelerating force muſt be always as the ſpace to be paſſed over during ſuch deſcent *;

* Let s be any ſpace to be paſſed over, z a part of that ſpace; and ſuppoſe that, in the time t, the moving body has paſſed over that part z, and has acquired a velocity v by the continued action of an accelerating force $= c.\ (s-z)$. Then will $c.\ (s-z).\ \dot{z} \div v = \dot{v}$, and conſequently $csz - \frac{1}{2}cz^2 = \frac{1}{2}v^2$, v being $= o$ when z is $= o$. Moreover $\dot{t}\ (= \dot{z} \div v)$ will be $= \dot{z} \div c^{\frac{1}{2}} \sqrt{2sz - z^2}$: and hence t is found $= \frac{1}{\sqrt{c}} \times$ circ. arc, radius 1, verſed ſine $z \div s$, conſequently taking z equal to s, we find the whole time of paſſing over the ſpace $s = \frac{1}{\sqrt{c}} \times$ quadrantal arc of the circle whoſe *radius is* 1. which, c being given, is *always the ſame, let s be what it* will.

that

that is, $\frac{1}{2}g.\ (\dot{x}-\dot{y}) \div \dot{z}$ must be $= c.\ (a-z)$, c being some variable quantity not yet known. Whence we have $g.\ (\dot{x}-\dot{y}) \div c = 2a\dot{z} - 2z\dot{z}$; and, by taking the fluents, we find $g.\ (\dot{x}-\dot{y}) \div c = 2az - z^2$.

Second. Let ABPERQN (Fig. 30, Pl. 1, in No. 1) be a semi-cycloid inverted, the diameter MH *plr* KN of whose generating circle is d: let AB be $= e$, BE $=$ EQ $= a$, HI $= b$, Hp $= x$, Kr $= y$, and BP $=$ QR $= z$; BH, Pp, EI, Rr. and QK being each parallel to the horizontal line AM. We shall then by the nature of the curve, have AN $= 2d$, HN $= (2d - e)^2 \div 4d$, Np $= (2d - e - z)^2 \div 4d$, NK $= (2d - e - 2a)^2 \div 4d$, N$r$ $= (2d - e - 2a + z)^2 \div 4d$, $(\overline{2d - e}|^2 - \overline{2d - e - z}|^2) \div 4d =$ HN $-$ Np $= x$, and $(\overline{2d - e - 2a + z}|^2 - \overline{2d - e - 2a}|^2) \div 4d =$ Nr $-$ NK $= y$. Hence, it appearing by subtraction that $x - y$ is $= (4az - 2z^2) \div 4d$, we have $g.\ (x - y) \div c = g.\ (4az - 2z^2) \div 4dc$; which, if c be $= g \div 2d$, will be $= 2az - z^2$, and the equation the same as that which we have deduced in the preceding article. It appears, therefore, that our cycloid is the curve required; and, the accelerating force of the ball P′ being $= g.\ (a - z) \div 2d$, the time of its descent from B′ to Q′ ($=$ twice the time of descent from B′ to E′) $= \sqrt{2d \div g} \times$ semicircle, rad. 1, which, d being given, will be the same, let B′E′ be what it will; and will be equal to twice the time of free descent, from B to N, in the same cycloid; or the limit of the time of vibration (in a circular arc) of a pendulum whose length is $2d$.

It is obvious that the consequence will be the same, if P′, P″ be similar, slender chains perfectly flexible. When P′ shall have descended from B′ to Q′, P″ having been drawn up from Q″ to B″, will begin

begin to descend from the last mentioned point and draw P′ upwards, so that a vibratory motion will ensue, which will be such, that, abstracting from friction, the time of vibration will be the same, from what point soever P′ may begin to move, and whatever may be the length of the line connecting the balls or chains. By means of which line a rod applied to a clock may be made to vibrate in any plane whatever: and only small parts of the cycloidal tubes being requisite, the mechanism may, in a little room, be so adapted, by taking the diameter *d* of a proper length, (agreeable to what is proved above,) that any given number of vibrations shall be performed in a given time.

The evolute of the cycloid being a similar cycloid, the balls (P′, P″) may be easily made to describe any cycloidal arcs by evolution; and, by substituting evolutes instead of tubes, the friction of the movement may be diminished; but it will then take up more room.

ARTICLE IX.

Observations on the fundamental Property of the Lever; with a Proof of the Principle assumed by Archimedes, *in his Demonstration. By the Rev. S. Vince, A. M. F. R. S.**

THE want of a demonstration of the property of the lever, upon clear and self-evident principles, has justly been considered as a great desideratum in the science of mechanics, as the most important parts of that branch of natural philosophy are founded upon it. *Archimedes* was, I believe, the first who attempted it. He supposes, that if two equal bodies be placed upon a lever, their effect to turn it about any point is the same as if they were placed

* Vide Philosophical Transactions, for 1794.

in

in the middle point between them. This propoſition is by no means ſelf-evident, and therefore the inveſtigation which is founded upon it has been rejected as imperfect. *Huygens* obſerves, that ſome mathematicians, not ſatisfied with the principle here taken for granted, have, by altering the form of the demonſtration, endeavoured to render its defects leſs ſenſible, but without ſucceſs. He then attempts a demonſtration of his own, in which he takes for granted, that if the ſame weight be removed to a greater diſtance from the fulcrum, the effect to turn about the lever will be greater; this is a principle by no means to be admitted, when we are ſuppoſed to be totally ignorant of the effects of weights upon a lever at different diſtances from the fulcrum. Moreover, if it were ſelf-evident, his demonſtration only holds when the lengths of the arms are commenſurable. Sir *I. Newton* has given a demonſtration, in which it is ſuppoſed, that if a given weight act in any direction, and any radii be drawn from the fulcrum to the line of direction, the effect to turn the lever will be the ſame on which ever of the radii it acts. But ſome of the moſt eminent mathematicians ſince his time have objected to this principle, as being far from ſelf-evident, and in conſequence thereof have attempted to demonſtrate the propoſition upon more clear and ſatisfactory principles. The demonſtration by *Mac Laurin*, as far as it goes, is certainly very ſatisfactory; but as he collects the truth of the propoſition only from induction, and has not extended it to the caſe where the arms are incommenſurable, his demonſtration is imperfect. The demonſtration given by *Dr. Hamilton*, in his Eſſays, depends upon this propoſition, that when a body is at reſt, and acted upon by three forces, they will be as the three ſides of a triangle parallel to the directions of the forces. Now this is true, when the three forces act at any point of a body; whereas, conſidering the lever as the body, the three

three forces act at different points, and therefore the principle, as applied by the author, is certainly not applicable. If in this demonſtration we ſuppoſe a plane body, in which the three forces act, inſtead of ſimply a lever, then, the three forces being actually directed to the ſame point of the body, the body would be at reſt. But in reaſoning from this to the caſe of the lever, the ſame difficulties would ariſe, as in the proof of Sir *I. Newton*. But admitting that all other objections could be removed, the demonſtration fails when any two of the forces are parallel Another demonſtration is founded upon this principle, that if two non-elaſtic bodies meet with equal quantities of motion, they will after impact, continue at reſt; and hence it is concluded, that if a lever which is in equilibrio be put in motion, the motions of the two bodies muſt be equal; and therefore the preſſures of theſe bodies upon the lever at reſt, to put it in motion, muſt be as their motions. Now, in the firſt place, this is comparing the effects of preſſure and motion, the relation of the meaſures of which, or whether they admit of any relation, we are totally unacquainted with. Moreover, they act under very different circumſtances; for, in the former caſe, the bodies acted immediately on each other, and in the latter they act by means of a lever, the properties of which we are ſuppoſed to be ignorant of. When forces act on a body, conſidered as a point, or directly againſt the ſame point of any body, we only eſtimate the effect of theſe forces to move the body out of its place, and no rotatory motion is either generated, or any cauſes to produce it, conſidered in the inveſtigation. When we, therefore, apply the ſame propoſition to inveſtigate the effect of the forces to generate a rotatory motion, we manifeſtly apply it to a caſe which is not contained in it, nor to which there is a ſingle principle applicable. The demonſtration given

given by *Mr. Landen*, in his Memoirs, is founded upon ſelf-evident principles, nor do I ſee any objections to his reaſoning upon them. But as his inveſtigation conſiſts of ſeveral caſes, and is, beſides, very long and tedious, ſomething more ſimple is ſtill much to be wiſhed for, proper to be introduced in an elementary treatiſe of mechanics, ſo as not to perplex the young ſtudent, either by the length of the demonſtration, or want of evidence in its principles. What I here propoſe to offer will, I hope, render the whole buſineſs not only very ſimple, but alſo perfectly ſatisfactory.

The demonſtration given by *Archimedes* would be very ſatisfactory and elegant, provided the principle on which it is founded could be clearly proved; viz. *that two equal powers at the extremities, or their ſum at the middle of a lever, would have equal effects to move it about any point.* Now, that the effects will be the ſame, ſo far as reſpects any *progreſſive* motion being communicated to the lever, when at liberty to move freely, is ſufficiently clear; but there is no evidence whatever that the effects will be the ſame to give the lever a *rotatory* motion about any point, becauſe a very different motion is then produced, and we are ſuppoſed to know nothing about the efficacy of a force at different diſtances from the fulcrum to produce ſuch a motion. Beſides, the two motions are not only different, but the *ſame* forces are known to produce *different* effects in the two caſes; for in the former caſe the two *equal* powers at the extremities of the arms produce *equal* effects in generating a *progreſſive* motion; but in the latter caſe they do *not* produce *equal* effects in generating a *rotatory* motion. We cannot therefore reaſon from one to the other. The principle, however, may be thus proved.

Let A C, (Figure 41, Plate 2.) be two equal bodies placed on a ſtraight lever, A P moveable about P; biſect AC in B, produce P A

to

to Q, and take B Q = B P, and ſuppoſe the end Q to be ſuſtained by a prop. Then as A and C are ſimilarly ſituated in reſpect to each end of the lever, that is, AP = CQ, and AQ = CP, the prop and fulcrum muſt bear equal parts of the whole weight; and therefore the prop at Q will be preſſed with a weight equal to A. Now take away the weights A and C, and put a weight at B equal to their ſum, and then the weight at B being equally diſtant from Q and P, the prop and fulcrum muſt ſuſtain equal parts of the whole weight, and therefore the prop will now alſo ſuſtain a weight equal to A. Hence if the prop Q be taken away, the moving force to turn the lever about P in both caſes muſt evidently be the ſame; therefore the effects of A and C upon the lever to turn it about any point are the ſame as when they are both placed in the middle point between them. And the ſame is manifeſtly true if A and C be placed without the fulcrum and prop. If therefore A C be a cylindrical lever of uniform denſity. its effect to turn itſelf about any point will be the ſame as if the whole were collected into the middle point B; which follows from what has been already proved, by conceiving the whole cylinder to be divided into an infinite number of laminæ perpendicular to its axis, of equal thickneſſes.

The principle therefore aſſumed by *Archimedes* is thus eſtabliſhed upon the moſt ſelf-evident principle, that is, that *equal* bodies at *equal* diſtances muſt produce *equal* effects; which is manifeſt from this conſideration, that when *all* the circumſtances in the cauſe are equal, the effects muſt be equal. Thus the whole demonſtration of *Archimedes* is rendered perfectly complete, and at the ſame time it is very ſhort and ſimple. The other part of the demonſtration we ſhall here inſert, for the uſe of thoſe who may not be acquainted with it.

Let

Let XY * be a cylinder, which bisect in A, on which point it would manifestly rest. Take any point Z, and bisect ZX in B, and ZY in C; then, from what has been proved, the effects of the two parts ZX, ZY to turn the lever about A is the same as if the weight of each part were collected into B and C respectively, which weights are manifestly as ZX, ZY, and which therefore conceive to be placed at B and C. Now $AB = AX - XB = \frac{1}{2} XY - \frac{1}{2} XZ = \frac{1}{2} YZ$; and $AC = AY - YC = \frac{1}{2} XY - \frac{1}{2} ZY = \frac{1}{2} XZ$; consequently $AB : AC :: \frac{1}{2} YZ : \frac{1}{2} XZ :: YZ : XZ ::$ the weight at C : the weight at B.

The property of the straight lever being thus established, every thing relative to the bent lever immediately follows.

ARTICLE X.

Investigations, founded on the Theory of Motion, for determining the Times of Vibration of Watch Balances. By George Atwood, Esq. F. R. S. ¶

INSTRUMENTS for measuring time by vibratory § motion were invented early in the sixteenth † century: the single pendulum ‡ had been known to afford a very exact measure of time long before this period; yet it appears, from the testimony

* Fig. 42, Plate 2. ¶ Vide Philosophical Transactions, 1794.

§ The ancients, as early as 140 years before CHRIST (probably much earlier) were acquainted with the use of wheel-work in constructing instruments for measuring time. "Denticuli alius alium impellentes, versationes modicas faciunt ac motiones," is the expression of VITRUVIUS in describing a machine, one of the principal uses of which was to indicate the hour of the day. Vibrations are no where mentioned or alluded to in the descriptions of the clocks constructed by the ancients. Dr. DERHAM on clock-work p. 86.

† About the year 1500, according to some accounts.

‡ TYCHO BRAHE is supposed to have used the pendulum in astronomical observations. RICCIOLUS, KIRCHER, MERSENNUS, and many others, are expressly mentioned by STURMIUS to have employed this method of measuring time.

of

of hiſtorical accounts, as well as other evidences, that the balance was univerſally adopted in the conſtruction of the firſt clocks and watches; nor was it till the year 1657 that Mr. *Huygens* united pendulums with clock-work.

The firſt eſſays of an invention, formed on principles at once new and complicated, we may ſuppoſe were imperfectly executed. In the watches of the earlier conſtructions, ſome of which are ſtill preſerved, the balance vibrated merely by the impulſes of the wheels, without other controul or regulation: the motion communicated to the balance by one impulſe continued till it was deſtroyed, partly by friction, and partly by a ſucceeding impulſe in the oppoſite direction; the vibrations muſt of courſe have been very unſteady and irregular.

Theſe imperfections were in a great meaſure remedied by Dr. *Hooke's* ingenious invention of applying a ſpiral ſpring to the balance *: the action of this ſpring on the balance of a watch is ſimilar to that of gravity on a pendulum: each kind of force has the effect of correcting the irregularities of impulſe and reſiſtance, which otherwiſe diſturb the iſochroniſm of the vibrations.

During the preſent century, various improvements have been made in the conſtruction of watches, principally by the artiſts of this country, to whoſe ingenuity and ſkill, aided and encouraged by public rewards, we muſt attribute the excellence of the modern watches and time-keepers, ſo highly valuable for their uſes in geography, navigation, and aſtronomy.

* Anno 1658.—An inſcription on a balance-ſpring watch, preſented to King Charles II. fixes the date of this invention to the year 1658. Dr. Derham relates, that he had ſeen the watch, on which the following inſcription was engraved: "Robert Hooke invent. 1658. T. Tompion fecit, 1675." Dr. Derham on Clock-work, p. 103.

The principles on which time-keepers are conſtructed, conſidered in a theoretical view, afford an intereſting ſubject of inveſtigation. It is always ſatisfactory to compare the motion of machines with the general law of mechanics, whenever friction and other irregular force are ſo far diminiſhed as to allow of a reference to theory, eſpecially if inferences, likely to be of practical uſe, may be derived from ſuch compariſon. In time-keepers, the irregular forces, both of impulſe and reſiſtance, are much diminiſhed by the exactneſs of form and dimenſion which is given to each part of the work; and they are further corrected by the maintaining power derived from the main ſpring; for whatever motion is loſt by the balance from reſiſtance of any kind, almoſt the ſame motion is communicated by the maintaining power, ſo as to continue the arc of vibration, as nearly as poſſible, of the ſame length.

In theſe machines, the real meaſure of time is the balance, all the other work ſerving only to continue the motion of the balance, and to indicate the time, as meaſured by its vibrations. The regularity of a time-keeper will therefore depend on that of the time in which the balance vibrates: to inveſtigate this time of vibration, from the ſeveral data or conditions on which it depends, is the object of the enſuing pages.

Let PMNS (Fig. 79, Pl. 5. No. 3.) repreſent the circumference of a watch balance, which vibrates by the action of a ſpiral * ſpring, on an axis paſſing through the centre C. Let ODBE be the circumference of a concentric circle, conſidered as fixed, to which the motion of the balance may be refered. In the circumference of this circle let any point O, be aſſumed, and when the balance is in its quieſcent

* In theſe inveſtigations it is indifferent whether the balance is ſuppoſed to vibrate by the action of a ſpiral or helical ſpring.

poſition,

poſition, ſuppoſe a line to be drawn through C and O, interſecting the circumference of the balance in the point A; the radius CA will be an index, by which the poſition of the balance, and its motion through any different arcs of vibration, will be truly defined. In the enſuing pages, the motion of the balance, and the motion of the index CA, will be uſed indifferently, as terms conveying the ſame meaning. Since the balance is in its quieſcent poſition when the index CA is directed to the fixed point O, on this account O is called the point of quieſcence of the balance, or balance ſpring, indicating the poſition when the balance is not impelled by the ſpring's elaſtic force either in one direction or the other. If the balance ſhould be turned through any angle OC B, the ſpiral ſpring being wound through the ſame angle, endeavours by its elaſtic force to reſtore itſelf; and when at liberty, impels the balance through the arc B O with an accelerated velocity till it arrives at the poſition O, where the force of acceleration ceaſes; with the velocity acquired at O, the balance proceeds in its vibration, deſcribing the arc O E with a retarded motion.

The elaſtic forces of the ſpring at equal diſtances on the oppoſite ſides of the point O, are aſſumed to be equal; it is alſo aſſumed that the effects of friction and other irregular reſiſtances which retard the motion of the balance, are compenſated by the maintaining power, ſo that the time of deſcribing the firſt arc of vibration B O by an accelerated motion ſhall be equal to the time of deſcribing the latter arc OE by a retarded motion, and that the entire arc of vibration BOE is biſected by the point O.

To render the conſtruction of Fig. 79. more diſtinct, the fixed circle ODBE is repreſented to be at a ſmall diſtance from the circumference of the balance, but is to be conſidered as coincident with it, ſo that the arc BO ſubtending the angle BCO, may be of the

 ſame

ſame length with an arc of the circumference of the balance which ſubtends the ſame angle BCO: on this principle CO or CA may be taken indifferently as the radius of the balance.

The determination of the time in which the balance vibrates, from the theory of motion, requires the following particulars to be known.

1ſt. The ſpring's elaſtic force, which impels the circumference of the balance when it is at a given angular diſtance OD (Fig. 79, Plate 5, No. 3.) from the quieſcent point O.

2dly. The law or ratio obſerved in the variation of the ſpring's force, while the balance is impelled from the extremity of the ſemiarc B to the point of quieſcence O, where all acceleration ceaſes.

3dly. The weight of the balance, including the parts which vibrate with it.

4thly. The radius of the balance CO, and the diſtance of the centre of gyration from the axis of motion CG.

5thly. The length of the ſemiarc BO.

Suppoſe the plane of the balance to be placed vertically, and let a weight P (Fig. 80, Plate 5, No. 3.) be applied, by means of a line ſuſpended freely from the circumference at T, to counterpoiſe the elaſtic force of the ſpring when the balance is wound through an angle from quieſcence OCD. This weight P (the weight of the line being allowed for) will be the force of the ſpiral ſpring which impels the circumference of the balance, when at the angular diſtance OD, from the quieſcent poſition.

It appears from many experiments, that the weights neceſſary to counterpoiſe a ſpiral ſpring's elaſtic force, when the balance is wound to the ſeveral diſtances from the quieſcent point, repreſented * by the arcs OG, OH, OI, (Fig. 81, Plate 5, No. 3.) &c.

* Berthoud Traité des Horloges marines, p. 49.

are

are nearly in the ratio of thofe feveral arcs. It alfo appears, that the fhape, the length, and number of turns of the fpiral may be fo adjufted to each other, that the forces of elafticity fhall be counterpoifed by weights which are in the precife ratio of the angular diftances from the quiefcent pofition, or, as it is fometimes expreffed, in the ratio of the fpring's tentions; at leaft as nearly as can be afcertained by experiment: this law of elaftic force is affumed in the fubfequent inveftigation.

To be continued.

ARTICLE XI.

LUCUBRATIONS IN SPHERICS.

By Mr. JOHN LOWRY.

PROP. X. THEOREM. Fig. 70, *Plate* 4.

IN any fpherical triangle, the rectangle contained under the fines of the bafe and perpendicular, is equal to the product of the fine of the fum of the fides, tangent of the arch bifecting the vertical angle, and the fine of half that angle.

Demon. Let S, C, Q, F reprefent the fines, and S′, C′, Q′, F′ the cofines of AG, BG, DG (arch bifecting the vertical angle AGB,) and ∠ AGB or BGD (half the vertical angle) refpectively, to the radius unity.

Then by p. 21$\frac{1}{4}$ Em. Trig. cot. ADG $= (SQ'F' - QS') \div SF$,

and cot. BDG $= (QC' - CQ'F') \div CF$,

hence $(SQ'F' - QS') \div SF = (QC' - CQ'F') \div CF$,

or $2CSQ'F' = (SC' + CS') \cdot Q$;

but by pr. 2. cor. 1. Sim. Trig. $(SC' + CS') \cdot Q = \text{f.}(AG + GB) \cdot Q$,

therefore $2CSQ'F' = \text{f.}(AG + GB) \cdot Q$,

or $CS = \text{f.}(AG + GB) \cdot (Q \div Q') \div 2F'$;

but $Q \div Q' = \text{f.}\,DG \div \text{cof.}\,DG = \tan.\,DG$,

 therefore

therefore $CS = s.(AG + GB) \cdot \tan. DG \div 2F'$;
but prop. II. $CS = s.AB \cdot s.GP \div s.\angle AGB, =$ (by sch. 1. prop. 2. book 1. Em. Trig.) $= s.AB \cdot s.GP \div 2FF$,
hence by eq. $s.AB \cdot s.GP \div 2FF' = s.(AG + GB) \cdot \tan. DG \div 2F$,
therefore $s.AB \cdot s.GP = s.(AG + GB) \cdot \tan. DG \cdot F$.

Q. E. D.

Cor. $F' = s.(AG + GB) \cdot \tan. DG \div 2CS$.

PROP. XI. THEOREM. Fig. 63, Plate 3.

If three arches be drawn from the three angular points of a spherical triangle ABC, to any point P; I say the product of the sines of the alternate angles will be equal.

Demon. From the triangles ACP, CBP, ABP, we have by trig. $s.CAP : s.CP :: s.ACP : s.AP$,
and $s.BCP : s.BP :: s.CBP : s.CP$,
and $s.ABP : s.AP :: s.BAP : s.BP$;
hence by compounding, it will be
$s.CAP \cdot s.BCP \cdot s.ABP : s.CP \cdot s.BP \cdot s.AP :: s.ACP \cdot s.CBP \cdot s.BAP : s.AP \cdot s.CP \cdot s.BP$,
but the consequents $s.CP \cdot s.BP \cdot s.AP$; $s.AP \cdot s.CP \cdot s.BP$ being equal, the antecedents
$s.CAP \cdot s.BCP \cdot s.ABP$; $s.ACP \cdot s.CBP \cdot s.BAP$
must be equal. *Q. E. D.*

Cor. 1. If two angles A, B, be bisected by the arches AP, PB, the third angle C, will also be bisected by the arch CP.

Cor. 2. Hence also, three perpendicular arches erected upon the middle of each side of a spherical triangle, all meet in one point.

PROP. XII. THEOREM. Fig. 71, Plate 4.

If four arches be drawn from the four angular points of any spherical trapezium ABCD, to any point P within it; I say the product of the sines of the alternate angles will be equal.

Demon. From the triangles APC, DPC, DPB, APB, it will be by Trig. $s.CAP : s.ACP :: s.CP : s.AP$,
and

and ſ. DCP : ſ. CDP :: ſ. PD : ſ. CP,
and ſ. PDB : ſ. DBP :: ſ. PB : ſ. PD,
and ſ. ABP : ſ. BAP :: ſ. AP : ſ. PB,
hence by comp. as ſ. CAP· ſ. DCP· ſ. PDB· ſ. ABP
is to ſ. ACP· ſ. CDP· ſ. DBP· ſ. BAP
ſo is ſ. CP· ſ. PD· ſ. PB· ſ. AP
to ſ. AP· ſ. CP· ſ.PD· ſ. PB ;

where the two laſt terms being equal, the two firſt terms muſt alſo be equal, that is,

ſ.CAP·ſ.DCP·ſ.PDB·ſ.ABP=ſ.ACP·ſ.CDP·ſ.DBP·ſ.BAP.

Q. E. D.

Cor. In like manner it may be proved, that if arches be drawn from the angular points of any ſpherical polygon whatever, to any point within it, the product of the ſines of the alternate angles will be equal.

PROP. XIII. THEOREM. Fig. 72, *Plate* 4.

If from the angles, at the baſe of any ſpherical triangle ABC, two arches APE, BPF, be drawn interſecting each other in the point P, and meeting the oppoſite ſides in E and F ; I ſay,

ſ. AP· ſ.BP : ſ. AC· ſ. CB :: ſ. EP· ſ. FP : ſ. EC· ſ. CF·

Demon. Through the points CP, let the arch CI, be drawn to meet the baſe AB in I ; alſo, draw the arches ED, FQ, making the angles at Q and D equal to the angles at I ;

then by trig. ſ. ACI : ſ. AI :: ſ. CIA : ſ. AC,
and ſ. ACI : ſ. FQ :: ſ. CQF : ſ. CF ;
by equality ſ. AI : ſ. FQ :: ſ. AC : ſ. CF ;
in like manner ſ. IB : ſ. DE :: ſ. CB : ſ. CE ;
therefore ſ.AI·ſ.IB.:ſ.FQ·ſ.DE::ſ.AC·ſ.CB:ſ.CF·ſ.CE.

Again by reaſon of the = ∠'s in the triangles AIP, PED ; IPB, PFQ,

we have ſ. AI : ſ. DE :: ſ.AP : ſ. PE,
and ſ. IB : ſ. QF :: ſ. BP : ſ. PF,
therefore ſ. AI·ſ.IB:ſ.DE·ſ.QF::ſ.AP·ſ.BP:ſ.PE·ſ.PF,
and by equal. ſ.AP·ſ.BP:ſ.AC·ſ.CB::ſ.PE·ſ.PF:ſ.CF·ſ.CE.

Q. E. D.

PROP.

PROP. XIV. THEOREM. Fig. 73, Plate 4.

Let ABC be any spherical triangle and BOD an arch bisecting the vertical angle; now if from E, the middle of the base AC, a perpendicular arch ED be drawn, intersecting the arch BOD in D, and from D let the arch DP be drawn perpendicular to the side AB;

I say the arch BP is equal to half the sum,
and the arch PA to half the difference
of the sides AB, BC.

Demon. From D demit the perpendicular arch DQ to meet the side BC produced in Q; join AD; DC,
then the right angled triangles AED, CED,
having AE = EC and ED common,
have also AD = DC;
also, the right angled triangles BPD, BQD,
having ∠ PBD = ∠ QBD and ∠ P = ∠ Q,
and BD being common,
will have DP = DQ;
and again, the right angled triangles APD, CQD;
having DP = DQ and AD = DC,
have likewise PA = CQ;
therefore AB + BC = AP + 2PB — CQ(AP) = 2PB;
and AC — CB = 2AP.

Q. E. D.

Cor. 1. The sum of the angles BAD, BCD = 180°.
For BCD + DCQ = 180°,
but BAD = DCQ,
therefore BCD + BAD = 180°.

Cor. 2. If the arch CI be drawn perpendicular to the side BC; then will ICP = half the difference of the angles ACB, CAB at the base,
for BCA—BAC = (BCA + ACD)—BAC + CAD or (ACD),
= BCD — BAD or DCQ,
= (90° + ICD) — (90° — ICD),
= 2 ICD.

PROP.

PROP. XV. PROBLEM.

Given the baſe, the difference of the ſides, and the difference of the angles at the baſe of a ſpherical triangle to conſtruct it.

Conſ. Let CQ (Fig. 73, Plate 4.) be half the difference of the ſides, and perpendicular thereto draw the arch QD, and from C draw the arch CD making the angle QCD = the compliment of half the difference of the angles at the baſe; about C as a pole with a diſtance equal to half the given baſe, deſcribe the leſſer circle LEK, and through D draw the great circle DE, to touch the leſſer one at E; produce CE till EA = CE, and join DA, then if the great circle APB be deſcribed to make the angle DAB = DCQ and meet QC produced in B; ABC will be the triangle required, as is evident from the laſt prop. and its corollaries.

PROP. XVI. THEOREM. Fig. 45, *Plate* 2.

If two ſides of a ſpherical triangle be given, the area will be a maximum when the triangle is inſcribed in a ſemicircle, the unknown ſide being the diameter.

Demon. Let ABG be a ſpherical triangle deſcribed in a ſemicircle, and draw the arch BD to the pole D; then ſince AD, DB, DG are all equal, the $\angle$ A = $\angle$ ABD and $\angle$G = DBG;
therefore $\angle$ A + $\angle$ G = $\angle$ B, and the triangle a maximum by qu. 709, Gent. Diary. *Q. E. D.*

Cor. 1. Hence it appears by reaſoning, as in Theo. 11. *Simpſon* on the Max. and Min. that the greateſt ſpherical polygon that can be contained under any propoſed number of given arches, and one other arch any how taken, will be when it may be inſcribed in a ſemicircle, the unknown arch being the diameter.

Cor. 2. If any ſpherical quadrilateral ABQS be inſcribed in a circle, the ſum of the oppoſite angles ABG,

ABG, ASG will be equal to the ſum of the oppoſite angles BAS, BGS.

Cor. 3. When AB = BG (Fig. 46, Plate 3.) the figure ABGS is a ſpherical ſquare having all its ſides and angles equal.

And by Trig. ſ. AG : ſ. BC :: ſ. ABG : ſ. BAG
:: 2ſ. GAB· coſ. GAB ÷ rad. : ſ. GAB
:: 2 coſ. GAB : radius.

Hence if radius be ſuppoſed = 1, the ſine of the ſide of any ſpherical ſquare will be equal to the ſine of the diameter of the circumſcribing circle divided by double the coſine of half the angle of the ſquare.

PROP. XVII. THEOREM. Fig. 61, *Plate* 3.

The difference between any two ſides of a ſpherical triangle is leſs than the third ſide.

Let ACB be a ſpherical triangle, the difference of any two ſides AC, CB is leſs than the third ſide AB.

Demon. Lay off CD = CB, join DB and produce CB to E;

then $\angle$ CDB = $\angle$ CBD or $\angle$ ADB = $\angle$ DBE,
but $\angle$ ABD is leſs than $\angle$ DBE or $\angle$ ADB,
therefore AD (= AC — CB) is leſs than AB.

Q. E. D.

PROP. XVIII. PROBLEM.

Given the baſe of a ſpherical triangle to conſtruct it, when its vertex falls in the arch of a great circle given by poſition, and the difference of its ſides is a maximum.

Conſ. Let AB (Fig. 65, Plate 4.) be the given baſe and QCI the given great circle; from B draw the circle PBD at right angles to QCI and make PD = PB; through the points A, D, deſcribe a great circle meeting QCI in C, and join CB; ſo will ACB be the triangle required.

Demon. Since AB is the given baſe and the vertex C falls in the arch of the given great circle QCI; there

remains

remains only to prove that AC — CB is a maximum, in order to which, take any other point I, and join AI, DI and IB, with great circles;

then becauſe DP = BP,
we have DC = BC and DI = BI,
therefore AC — CB = AC — CD = AD,
and AI — IB = AI — ID;
but by the laſt prop. AI — ID is leſs than AD,
therefore AC — CB is a maximum.

Q. E. D.

PROP. XIX. THEOREM. Fig. 63, 64, *Plate* 3, 4.

If CE, CF and CD, be three great circles of the ſphere interſecting in C, and AB, AI, two other great circles drawn from any point A in CD, interſecting CE and CF in B and I, ſuch that the arch AB = arch AI; then if from any other point in CD as P, two more great circles PS, PQ be drawn to make ∠ CQP = ∠ CBA and ∠ CSP = ∠ CIA; I ſay the arch PQ will be equal to the arch PS.

Demon. By trig. ſ. AB : ſ. BCA : : ſ. AC : ſ. CBA,
and ſ. QP : ſ. QCP : : ſ. CP : ſ. CQP,
but QCP = BCA and CQP = CBA,
hence by equality ſ. AB : ſ. QP : : ſ. AC : ſ. CP;
in like manner ſ. IA : ſ. SP : : ſ. AC : ſ. CP,
therefore by equality ſ. AB : ſ. QP : : ſ. IA : ſ. SP;
but the antecedents ſ. AB, ſ. IA are equal,
therefore the conſequents ſ. QP, ſ. SP muſt be equal.

Q. E. D.

PROP. XX. THEOREM. Fig. 66, *Plate* 4.

Let P be a given point, and AB, AC two great circles given in poſition; it is required to draw another great circle PQ to interſect AC in Q, ſo that if the arch QL be drawn to make a given angle with AB, the ſum of the arches PQ, QL may be the leaſt poſſible.

Conſ.

Conſ. From any point as b in AC, draw the great circle BD to make $\angle$ BDA $=$ the given one; and with the center b and diſtance bD deſcribe the leſſer circle DF, and from A draw the great circle ARF to touch it at F; from the given point P draw the great circle PQR perpendicular to AF, interſecting AC in Q; then if QL be drawn making the given angle with AB; I ſay the arches PQ, QL will be thoſe required.

Demon. Draw any other great circle PHS, and demit the perpendicular arch HE; alſo, draw the arch HI making the $\angle$ AIH $= \angle$ ALQ, and through the points PE, deſcribe a great circle;

now by conſ. arch bD $=$ arch bF,
alſo by laſt prop. arch HE $=$ arch HI,
and arch QR $=$ arch QL,
therefore arch PH $+$ arch HI $=$ arch PH $+$ arch HE,
and arch PQ $+$ arch QL $=$ arch PR;
but arch PR is leſs than the arch PE,
and arch PE is leſs than arch PH $+$ arch HE,
therefore the arch PR or the ſum of the arches PQ, QL is leſs than the ſum of the arches PH, HE.

Q. E. D.

Obſervation 1. When the arch PR is greater than a quadrant, the ſum of the arches PQ, QL is a maximum inſtead of a minimum, for then the arch PR is greater than the arch PS.

2. When the given point is ſituated between the given great circles, the *conſtruction* is nearly the ſame as above.

3. When the given point is on the other ſide of the arch AC, as in (Fig. 44, Plate 2.) then the difference of the arches PQ, QL is a minimum. For then the arch PE, the difference of the arches PH, HI is greater than the arch PR the difference of the arches PQ, QL, unleſs the arch PR be greater than a quadrant, and then the ſum of the arches PQ, Q L is a maximum.

PROP.

PROP. XXI. PROBLEM.

To deſcribe the circumference of a leſſer circle through a given point to touch two great circles given by poſition.

Conſ. Let P (Fig. 67, Plate 4.) be the given point, and BAQ, BOQ the given great circles; through P deſcribe the great circle BPQ, and biſect the angle OBA with the great circle BCQ, from any point in which draw the arch DI at right angles to the great circle BOQ; to the great circle BPQ apply the arch DE = arch DI; from P draw the arch PC making the ∠ BPC = ∠ BED; with the centre C and diſtance CP deſcribe a circle, and the thing will be done.

Demon. Let the arches CO, CA be drawn perpendicular to the great circles BOQ, BAQ reſpectively;

then by *conſ.* arch DE = arch DI,
and by prop. xix. arch CO = arch CP,
therefore the circles touch at O.

Again, the right angled triangles BOC, BAC having the arch BC common to both, and the angles at B equal in each, are equal and ſimilar in every reſpect;

wherefore arch CA = arch CO = arch CP,
conſequently the circles touch at A.

Q.E.D.

Obſervation. There are two circles that will anſwer the conditions of the problem: for two arches DE, De, may be applied from the point D = to the arch DI.

PROP. XXII. THEOREM. Fig. 68, *Plate* 4.

If a leſſer circle ABC, touch either a greater or a leſſer circle ADQ as at A, and another circle

BCDQ be deſcribed to interſect the former circles as at B, C; D, Q. Then I ſay if the chords QD, BC be produced they will meet the tangent drawn from the point of contact A, in the ſame point P.

Demon. The points B, C, D, Q being in the ſame plane, the chords BC, DQ produced will meet in ſome point P; again, the points A, B, C being in the ſame plane, the chord BC and the tangent AP will alſo meet in ſome point.

By the nature of the circles QDBC, ABC,
we have rect. $DP \cdot PQ = $ rect. $CP \cdot PB$,
and rect. $CP \cdot PB = PA^2$,
therefore by equality rect. $DP \cdot PQ = PA^2$;

hence AP is a common tangent to both the circles, and conſequently the chords DQ, BC produced meet the tangent AP in the ſame point P.

Q. E. D.

PROP. XXIII. PROBLEM.

To deſcribe the circumference of a leſſer circle through a given point, to touch a given great circle and have its centre in the arch of a great circle given by poſition.

Conſ. Let P (Fig. 67, Plate 4.) be the given point, BOQ the given great circle, and BCQ the great circle given by poſition, in which the centre of the required circle muſt be ſituated; from B draw the great circle BAQ making the $\angle$ QBA $= \angle$ QBO; Then by prop. xxi. deſcribe a circle APO to paſs through the point P and touch the great circles BOQ, BAQ, and the thing will be done.

This requires no particular *demonſtration.*

Otherwiſe,

Conſ. Let C (Fig. 68, Plate 4.) be the given point, RS the great circle given by poſition, and ADQ the other great circle; through the point C deſcribe

a leſſer

a leſſer circle having its centre in the arch of the great circle RS, and interſecting the given great circle ADQ in D and Q; make the arch IB = arch IC, and draw the chords BC, QD to interſect in P; from P draw the tangent PA to touch the circle ADQ in A; then through the points A, B, C deſcribe a circle, and the buſineſs will be finiſhed.

The reaſon of the above *Conſ.* will appear evident from the laſt prop.

PROP. XXIV. PROBLEM.

From a given point to draw a great circle, to interſect two given parallel circles, ſo as to have a given arch intercepted by their peripheries.

Conſ. Let P (Fig. 69, Plate 4.) be the given point; BCQ, AFS the given circles; apply the arch BS = the given arch, and on it continued, demit the perpendicular arch OR from the centre O; then with the common centre O and diſtance OR deſcribe the leſſer circle RI and from P draw the great circle PAIFQ to touch at I; and the thing is done.

Demon. By parallel circles, arch BR = arch IQ,
and arch SR = arch IF,
therefore arch FQ = arch BS = given arch.

Q. E. D.

Obſervation 1. The intercepted part is a maximum when the arch OR = arch OS; and a minimum when the required circle paſſes through the centre O.

2. If the great PAFQ had been required to make a given angle with a given great circle, inſtead of paſſing through a given point; then if the circle RI be deſcribed as before, the problem will be to deſcribe the circle PAFQ to touch it at I and make a given angle with a given great circle, which is elegantly done by Mr. *Wales* at qu. 39, Miſcell. Scien. Curioſa.

ARTICLE XII.

Of the ELLIPSIS and HYPERBOLA, from Mr. LANDEN's MEMOIRS.

SOME of the theorems given by mathematicians for the calculation of fluents by means of elliptic and hyperbolic arcs, requiring in the application thereof, the difference to be taken between an arc of a hyperbola and its tangent; and such difference being not directly attainable when such arc and its tangent both become infinite, as they will do when the *whole* fluent is wanted, although such fluent be at the same time finite; those thorems therefore in that case fail, a computation thereby being then impracticable without some farther help.

The supplying that defect I considered as a point of some importance in geometry, and therefore I earnestly wished and endeavoured to accomplish that business; my aim being to ascertain, by means of such arcs as above mentioned, the *limit* of the difference between the hyperbolic arc and its tangent, whilst the point of contact is supposed to be carried to an infinite distance from the vertex of the curve, seeing that, by the help of that *limit*, the computation would be rendered practicable in the case wherein, without such help, the before mentioned theorems fail. The result of my endeavours respecting that point appears in this memoir, which, amongst other matters, contains the investigation of a general theorem for finding the length of any arc of any conic hyperbola by means of two elliptic arcs. A discovery (first published by me in the *Philosophical Transactions* for 1775,) whereby we are enabled to bring out very elegant conclusions in many interesting enquiries, as well mechanical as purely geometrical.

1. Suppose the curve ADEF (Fig. 34, Plate 2.) be a conic *hyperbola*, whose semi-transverse axis AC is

is $= m$, and semi-conjugate $= n$. Let CP, perpendicular to the tangent DP be called p; and put $f = (m^2 - n^2) \div 2m$, $z = p^2 \div m$. Then will DP—AD be $=$ the fluent of $-\frac{1}{2}m^{\frac{1}{2}}z^{\frac{1}{2}}\dot{z} \div \sqrt{n^2 + 2fz - z^2}$, p and z being each $= m$ when AD is $= o$. For, denoting the semidiameter CD by r, and its semiconjugate diameter by s, we have (by the nature of the curve) $r^2 - s^2 = m^2 - n^2 = 2fm$, and $ps = mn$. Whence $s^2 = r^2 - 2fm = m^2n^2 \div p^2 = mn^2 \div z$; and consequently $r^2 = (2fmz + mn^2) \div z$, and DP $= \sqrt{r^2 - p^2} = \sqrt{mn^2 + 2fmz - mz^2} \div \sqrt{z}$. Hence the fluxion of DP is found $= -(m^{\frac{1}{2}}n^2\dot{z} + m^{\frac{1}{2}}z^2\dot{z}) \div 2z^{\frac{3}{2}}\sqrt{n^2 + 2fz - z^2}$. Now it is obvious that the fluxion of the curve AD is to $\dot{r}$ as r to $\sqrt{r^2 - p^2}$: therefore the fluxion of AD is $= r\dot{r} \div \sqrt{r^2 - p^2}$, which by substitution appears to be $= -m^{\frac{1}{2}}n^2\dot{z} \div 2z^{\frac{3}{2}}\sqrt{n^2 + 2fz - z^2}$. Consequeutly the difference of the fluxions of DP and AD is $= -\frac{1}{2}m^{\frac{1}{2}}z^{\frac{1}{2}}\dot{z} \div \sqrt{n^2 + 2fz - z^2}$.

2. Suppose the curve adefg (Fig. 35, Plate 2.) to be a quadrant of an *ellipsis*, whose semi-transverse axis cg is $= \sqrt{m^2 + n^2}$, and semi-conjugate ac $= n$. Let ct be perpendicular to the tangent dt, and let the abscissa cp be $= n\sqrt{z \div m}$. Then will the said tangent dt be $= m\sqrt{(mz - z^2) \div (n^2 + mz)}$; and the fluxion thereof will be found $= \frac{1}{2}mn^2z^{-\frac{1}{2}}\dot{z}\sqrt{m - z} \div (n^2 + mz)^{\frac{3}{2}} - \frac{1}{2}m^{\frac{1}{2}}z^{\frac{1}{2}}\dot{z} \div \sqrt{n^2 + 2fz - z^2}$.

3. In the expression $y^{q-1}\dot{y} \div ((a + by)^r . (c + dy)^s)$, let $(c + dy) \div (a + by)$ be supposed $= z$. Then will

 (az

$(az - c) \div (d - bz)$ be $= y$, and the propofed expreffion will be $= (ad - bc)^{1 - r - s} z^{-s} \dot{z} \div \left((az - c)^{1-q} \cdot (d - bz)^{1 + q - r - s}\right)$.

4. Taking, in the laft article, r and s each $= \frac{1}{2}$, $q = \frac{3}{2}$, $a = -d = n^2 \div m$, $b = 1$, and $c = n^2$, we have $y^{\frac{1}{2}}\dot{y} \div \left(\sqrt{(n^2 \div m)} + y \times \sqrt{n^2 - n^2 y \div m}\right)$ $\left(= m^{\frac{1}{2}} n^{-1} y^{\frac{1}{2}} \dot{y} \div \sqrt{n^2 + 2fy - y^2}\right) = - mnz^{-\frac{1}{2}} \dot{z} \sqrt{m - z} \div (n^2 + mz)^{\frac{3}{2}}$. It appears therefore, that, y being $= n^2 . (m - z) \div (n^2 + mz)$, $-\frac{1}{2}m^{\frac{1}{2}} y^{\frac{1}{2}}\dot{y} \div \sqrt{n^2 + 2fy - y^2} - \frac{1}{2}m^{\frac{1}{2}} z^{\frac{1}{2}} \dot{z} \div \sqrt{n^2 + 2fz - z^2}$ is $= \frac{1}{2}mn^2 z^{-\frac{1}{2}} \dot{z} \sqrt{m - z} \div (n^2 + mz)^{\frac{3}{2}} - \frac{1}{2}m^{\frac{1}{2}} z^{\frac{1}{2}} \dot{z} \div \sqrt{n^2 + 2fz - z^2}$; which, by article 2, is $=$ the *fluxion of the tangent* dt.

Confequetly, taking the fluents by article 1, and correcting them properly, we find

$$DP - AD + FR - AF = L + dt.$$

CP (Fig. 34. Pl. 2.) being $= m^{\frac{1}{2}} z^{\frac{1}{2}}$; cp (Fig. 35, Pl. 2.) $= n\sqrt{z \div m}$; CR, perpendicular to the tangent FR $= m^{\frac{1}{2}} y^{\frac{1}{2}}$; DP—AD $=$ the *fluent* of $-\frac{1}{2}m^{\frac{1}{2}} z^{\frac{1}{2}} \dot{z} \div \sqrt{n^2 + 2fz - z^2}$; FR—AF $=$ the *fluent* of $-\frac{1}{2}m^{\frac{1}{2}} y^{\frac{1}{2}} \dot{y} \div \sqrt{n^2 + 2fy - y^2}$; and L the *limit* to which the difference DP — AD, or FR—AF, approaches upon carrying the point D, or E, from the vertex A *ad infinitum*.

5. Suppofe y equal to z, and that the points D and F will then coincide in E, the points d and p being at the fame time in e and q refpectively. Then cv being perpendicular to the tangent ev, that tangent

gent will be a *maximum* and equal to $cg - ac =$ $\sqrt{m^2+n^2} - n$; the tangent EQ (in the hyperbola) will be $= \sqrt{m^2+n^2}$; the abſciſſa $BC = m \sqrt{1+(n \div \sqrt{m^2+n^2}}$; the ordinate $BE = n \sqrt{n \div \sqrt{m^2+n^2}}$; and it appears, that L is $= 2EQ - 2AE - ev = n + \sqrt{m^2+n^2} - 2AE$. Thus the *limit* which I propoſed to aſcertain is inveſtigated, *m* and *n* being any right lines whatever. Another expreſſion for ſuch *limit* will be found in a ſubſequent article in this memoir.

ARTICLE XIII.

Of finding the ſums of certain ſeries by Mr. Stirling's *differential method, by Mr.* J. Mabbot, Mancheſter.

LET the ſeries propoſed to be ſummed be

$$\frac{1}{2 \cdot 6} + \frac{1}{4 \cdot 8} + \frac{1}{6 \cdot 10} + \&c.$$

Here $T = \dfrac{1}{2z \cdot 2z+4} = \dfrac{1}{4z \cdot z+1} - \dfrac{1}{4z \cdot z+1 \cdot z+2}$, the values of z being 1, 2, 3, &c.

and $S = \dfrac{1}{4z} - \dfrac{1}{8z \cdot z+1} = \dfrac{2z^2+z}{8z^3+8z^2} = \dfrac{2z+1}{8z^2+8z} = \dfrac{3}{16}$, the ſum required, when z is taken $= 1$.

2. Let the ſame ſeries be propoſed to be ſummed when the ſigns change alternately, *i. e.*

$$\frac{1}{2 \cdot 6} - \frac{1}{4 \cdot 8} + \frac{1}{6 \cdot 10} - \&c. \text{ Here}$$

$$T = \overline{-1}|^{z-1} \times \frac{1}{2z \cdot 2z+4} = \overline{-1}|^{z-1} \times \left(\frac{1}{4z \cdot z+1} - \frac{1}{4z \cdot z+1 \cdot z+2}\right)$$

x being $= -1$, and

S =

$S = \overline{-1}|^{z-1} \times \frac{1}{8z\cdot z+1} = \frac{1}{16}$ when z is taken $= 1$;

B being $= \frac{1}{8}$, A, C, D &c. each $= 0$.

3. Required the ſum of the infinite ſeries?

$$\frac{1}{4\cdot 8} - \frac{1}{6\cdot 10} + \frac{1}{8\cdot 12} - \text{\&c.}$$

Here $T = \overline{-1}|^{z-2} \times \frac{1}{4z\cdot z+2} = \overline{-1}|^{z-2} \times \left(\frac{1}{4z\cdot z+1} - \frac{1}{4z\cdot z+1\cdot z+2}\right)$

x being $= -1$ and

$S = \overline{-1}|^{z-2} \times \frac{1}{8z\cdot z+1} = \frac{1}{48}$, when z is taken $= 2$;

B being $= \frac{1}{8}$, A,C,D &c. each $= 0$.

4. Required the ſum of the infinite ſeries?

$\frac{1}{5\cdot 12} + \frac{1}{10\cdot 15} + \frac{1}{15\cdot 18} +$ &c. Here $T =$

$$\frac{1}{15z\cdot z+3} = \frac{1}{15z\cdot z+1} - \frac{2}{15z\cdot z+1\cdot z+2} + \frac{2}{15z\cdot z+1\cdot z+2\cdot z+3}$$

the values of z being 1, 2, 3, 4 &c. and $S =$

$$\frac{1}{15z} - \frac{1}{15z\cdot z+1} + \frac{2}{45z\cdot z+1\cdot z+2} = \frac{3z^2+6z+2}{45z\cdot z+1\cdot z+2} = \frac{11}{270}$$

when z is taken $= 1$.

5. Required the ſum of the ſeries?

$$\frac{1}{1\cdot 2\cdot 4} + \frac{1}{2\cdot 3\cdot 5} + \frac{1}{3\cdot 4\cdot 6} + \text{\&c.}$$

Here $T = \frac{1}{z\cdot z+1\cdot z+3} = \frac{1}{z\cdot z+1\cdot z+2} - \frac{1}{z\cdot z+1\cdot z+2\cdot z+3}$

and $S = \frac{1}{2z\cdot z+1} - \frac{1}{3z\cdot z+1\cdot z+2} = \frac{3z+4}{6z\cdot z+1\cdot z+2} = \frac{7}{36}$

when z is taken $= 1$.

6. Let the ſeries to be ſummed be

$$\frac{1}{1\cdot 2\cdot 3} + \frac{1}{2\cdot 3\cdot 4} + \frac{1}{3\cdot 4\cdot 5} + \text{\&c.}$$

Here $T = \frac{1}{z\cdot z+1\cdot z+2}$

and

and $S = \frac{1}{2z\cdot\overline{z+1}} = \frac{1}{4}$, when z is taken $= 1$.

7. Required the ſum of the ſeries ?

$$\frac{1}{2\cdot6\cdot10} - \frac{1}{4\cdot8\cdot12} + \frac{1}{6\cdot10\cdot14} - \&c.$$

Here $T = \overline{-1}\rceil^{z-1} \times \frac{1}{8z\cdot\overline{z+2}\cdot\overline{z+4}} = \overline{-1}\rceil^{z-1} \times$

$$\left(\frac{1}{8z\cdot\overline{z+1}\cdot\overline{z+2}} - \frac{3}{8z\cdot\overline{z+1}\cdot\overline{z+2}\cdot\overline{z+3}} + \frac{3}{8z\cdot\overline{z+1}\cdot\overline{z+2}\cdot\overline{z+3}\cdot\overline{z+4}}\right)$$

the values of z being 1, 2, 3, 4 &c. and $S =$

$$\overline{-1}\rceil^{z-1} \times \left(\frac{1}{16z\cdot\overline{z+1}\cdot\overline{z+2}} - \frac{3}{32z\cdot\overline{z+1}\cdot\overline{z+2}\cdot\overline{z+3}}\right)$$

$$= \overline{-1}\rceil^{z-1} \times \frac{2z+3}{32z\cdot\overline{z+1}\cdot\overline{z+2}\cdot\overline{z+3}} = \frac{2+3}{32\cdot2\cdot3\cdot4} = \frac{5}{768}$$

when z is taken $= 1$.

C being $= \frac{1}{16}$, $D = -\frac{3}{32}$, A, B, E &c. each $= 0$.

8. Let the ſeries propoſed be

$$\frac{1}{1\cdot9\cdot10} - \frac{1}{2\cdot12\cdot12} + \frac{1}{3\cdot15\cdot14} - \&c.$$

Here $T = \overline{-1}\rceil^{z-1} \times \frac{1}{z\cdot\overline{3z+6}\cdot\overline{2z+8}} = \overline{-1}\rceil^{z-1} \times$

$$\left(\frac{1}{6z\cdot\overline{z+1}\cdot\overline{z+2}} - \frac{3}{6z\cdot\overline{z+1}\cdot\overline{z+2}\cdot\overline{z+3}} + \frac{3}{6z\cdot\overline{z+1}\cdot\overline{z+2}\cdot\overline{z+3}\cdot\overline{z+4}}\right)$$

z being $= 1$, and $S = \overline{-1}\rceil^{z-1} \times \left(\frac{1}{12z\cdot\overline{z+1}\cdot\overline{z+2}}\right.$

$$\left. - \frac{1}{8z\cdot\overline{z+1}\cdot\overline{z+2}\cdot\overline{z+3}}\right) = \overline{-1}\rceil^{z-1} \times \frac{2z+3}{24z\cdot\overline{z+1}\cdot\overline{z+2}\cdot\overline{z+3}} = \frac{5}{24}.$$

when z is taken $= 1$, C being $= \frac{1}{12}$, $D = -\frac{1}{8}$,

A, B, E &c. each equal 0.

9. Let

9. Let the ſeries to be ſummed be

$$\frac{1}{2\cdot6\cdot8}+\frac{1}{3\cdot8\cdot10}+\frac{1}{4\cdot10\cdot12}+\ \&c.\quad \text{Here}$$

$$T=\frac{1}{z+1\cdot2z+4\cdot2z+6}=\frac{1}{4z\cdot z+1\cdot z+2}-\frac{3}{4z\cdot z+1\cdot z+2\cdot z+3},$$

the values of z being 1, 2, 3 &c. and

$$S=\frac{1}{8z\cdot z+1}-\frac{1}{4z\cdot z+1\cdot z+2}=\frac{z}{8z\cdot z+1\cdot z+2}=\frac{1}{48},$$

when z equal 1.

10. Let the ſeries propoſed be

$$\frac{1}{1\cdot2\cdot3\cdot4}+\frac{1}{2\cdot3\cdot4\cdot5}+\frac{1}{3\cdot4\cdot5\cdot6}+\ \&c.\quad \text{Here}$$

$T=\frac{1}{z\cdot z+1\cdot z+2\cdot z+3}$; the values of z being 1, 2, 3 &c.

and $S=\frac{1}{3z\cdot z+1\cdot z+2}=\frac{1}{18}$ when z is taken $=1$.

11. Required the ſum of the infinite ſeries?

$$\frac{1}{2\cdot4\cdot5\cdot6}+\frac{1}{3\cdot5\cdot6\cdot7}+\frac{1}{4\cdot6\cdot7\cdot8}+\ \&c.$$

$$\text{Here } T=\frac{1}{z+1\cdot z+3\cdot z+4\cdot z+5}=\frac{1}{z\cdot z+1\cdot z+2\cdot z+3}$$

$$-\frac{7}{z\cdot z+1\cdot z+2\cdot z+3\cdot z+4}+\frac{15}{z\cdot z+1\cdot z+2\cdot z+3\cdot z+4\cdot z+5}$$

the values of z being 1, 2, 3 &c. and $S=$

$$\frac{1}{3z\cdot z+1\cdot z+2}-\frac{7}{4z\cdot z+1\cdot z+2\cdot z+3}+\frac{3}{z\cdot z+1\cdot z+2\cdot z+3\cdot z+4}$$

$$=\frac{4z^2+7z}{12z\cdot z+1\cdot z+2\cdot z+3\cdot z+4}=\frac{4z+7}{82\cdot z+1\cdot z+2\cdot z+3\cdot z+4}$$

$$=\frac{11}{1440},\ \text{when } z \text{ is taken } =1.$$

12. Let the ſeries propoſed be

$$\frac{5}{1\cdot2\cdot3}+\frac{6}{2\cdot3\cdot4}+\frac{7}{3\cdot4\cdot5}+\ \&c.$$

Here

Here $T = \frac{z+4}{z\cdot\overline{z+1}\cdot\overline{z+2}} = \frac{1}{z\cdot\overline{z+1}} + \frac{2}{z\cdot\overline{z+1}\cdot\overline{z+2}}$

the values of z being 1, 2, 3 &c.

and $S = \frac{1}{z} + \frac{1}{z\cdot\overline{z+1}} = \frac{z+2}{z\cdot\overline{z+1}} = \frac{3}{2}$ when z is $= 1$.

13. Let the series proposed be

$$\frac{1}{3\cdot 6\cdot 28} + \frac{4}{6\cdot 8\cdot 35} + \frac{7}{9\cdot 10\cdot 42} + \&c.$$

$$\text{Here } T = \frac{3z-2}{3z\cdot\overline{2z+4}\cdot\overline{7z+21}}$$

$$= \frac{1}{14z\cdot\overline{z+1}} - \frac{1}{3z\cdot\overline{z+1}\cdot\overline{z+2}} + \frac{11}{21z\cdot\overline{z+1}\cdot\overline{z+2}\cdot\overline{z+3}}$$

$$\text{and } S = \frac{1}{14z} - \frac{1}{6z\cdot\overline{z+1}} + \frac{11}{63z\cdot\overline{z+1}\cdot\overline{z+2}}$$

$$= \frac{9z^2+6z^2-2}{126z\cdot\overline{z+1}\cdot\overline{z+2}} = \frac{13}{756}, \text{ when } z \text{ is } = 1.$$

14. Let the series to be summed be

$$\frac{2\cdot 5}{1\cdot 6\cdot 15\cdot 21} - \frac{3\cdot 8}{2\cdot 8\cdot 18\cdot 24} + \frac{4\cdot 11}{3\cdot 10\cdot 21\cdot 27} - \&c.$$

$$\text{Here } T = \overline{-1}\rceil^{z-1} \times \frac{\overline{z+1}.\overline{3z+2}}{z\cdot\overline{2z+4}\cdot\overline{3z+12}\cdot\overline{3z+18}}$$

$$= \overline{-1}\rceil^{z-1} \times \left(\frac{1}{6z\cdot\overline{z+1}} - \frac{14}{9z\cdot\overline{z+1}\cdot\overline{z+2}} + \frac{155}{18z\cdot\overline{z+1}\cdot\overline{z+2}\cdot\overline{z+3}}\right.$$

$$- \frac{185}{6z\cdot\overline{z+1}\cdot\overline{z+2}\cdot\overline{z+3}\cdot\overline{z+4}} + \frac{200}{3z\cdot\overline{z+1}\cdot\overline{z+2}\cdot\overline{z+3}\cdot\overline{z+4}\cdot\overline{z+5}}$$

$$\left. - \frac{200}{3z\cdot\overline{z+1}\cdot\overline{z+2}\cdot\overline{z+3}\cdot\overline{z+4}\cdot\overline{z+5}\cdot\overline{z+6}}\right),$$

x being $= 1$ and the values of z being 1, 2, 3 &c.

$$\text{and } S = \overline{-1}\rceil^{z-1} \times \left(\frac{1}{12z\cdot\overline{z+1}} - \frac{75}{108z\cdot\overline{z+1}\cdot\overline{z+2}}\right.$$

$$+ \frac{235}{72z\cdot\overline{z+1}\cdot\overline{z+2}\cdot\overline{z+3}} - \frac{160}{18z\cdot\overline{z+1}\cdot\overline{z+2}\cdot\overline{z+3}\cdot\overline{z+4}}$$

+

[illegible]

$\therefore$ [illegible] $= \frac{1}{12}$, $C = -\frac{75}{[illegible]}$, $D = \frac{[illegible]}{7[illegible]}$, $E = -$ [illegible]

$F = \frac{192}{9}$. G, H and all the rest $= 0$.

15. Required the value of series?

$$\frac{3\cdot 8}{3\cdot 6\cdot 12\cdot 15} + \frac{5\cdot 13}{6\cdot 9\cdot 15\cdot 18} + \frac{7\cdot 18}{9\cdot 12\cdot 18\cdot 21} + \&c.$$

Here $T = \frac{2z+1\cdot 5z+3}{3z\cdot 3z+3\cdot 3z+9\cdot 3z+12} = \frac{10}{54 z\cdot z+1}$

$$-\frac{69}{54 z\cdot z+1\cdot z+2} + \frac{237}{54 z\cdot z+1\cdot z+2\cdot z+3}$$

$$-\frac{357}{54 z\cdot z+1\cdot z+2\cdot z+3\cdot z+4} \text{ and } S = \frac{10}{54 z} -$$

$$\frac{69}{108 z\cdot z+1} + \frac{237}{162 z\cdot z+1\cdot z+2} - \frac{357}{216 z\cdot z+1\cdot z+2\cdot z+3} =$$

$$\frac{40z^3+102z^2+66z+3}{216 z\cdot z+1\cdot z+2\cdot z+3} = \frac{211}{5184} \text{ when } z = 1.$$

16. The series to be summed being

$$\frac{1}{3\cdot 4\cdot 6\cdot 7} - \frac{1}{4\cdot 5\cdot 7\cdot 8} + \frac{1}{5\cdot 6\cdot 8\cdot 9} - \&c.$$

We shall have $T = \overline{-1}|^{z-1} \times \frac{1}{z+2\cdot z+3\cdot z+4\cdot z+5}$

$$= \overline{-1}|^{z-1} \times \left(\frac{1}{z\cdot z+1\cdot z+2\cdot z+3} - \frac{10}{z\cdot z+1\cdot z+2\cdot z+3\cdot z+4}\right.$$

$$+ \frac{40}{z\cdot z+1\cdot z+2\cdot z+3\cdot z+4\cdot z+5}$$

$$\left. - \frac{60}{z\cdot z+1\cdot z+2\cdot z+3\cdot z+4\cdot z+5\cdot z+6}\right)$$

x being

[illegible] $g = -1$; and $S = \overline{-1}|^{p-1} \times \left(\frac{1}{2 \cdot z \cdot \overline{z+1} \cdot \overline{z+2} \cdot \overline{z+3}} \right.$ [illegible] $\left. \frac{4}{z \cdot \overline{z+1} \cdot \overline{z+2} \cdot \overline{z+3} \cdot \overline{z+4}} + \frac{10}{z \cdot \overline{z+1} \cdot \overline{z+2} \cdot \overline{z+3} \cdot \overline{z+4} \cdot \overline{z+5}} \right) =$ [illegible] $\times \frac{z^2 + z}{2 \cdot z \cdot \overline{z+1} \cdot \overline{z+2} \cdot \overline{z+3} \cdot \overline{z+4} \cdot \overline{z+5}} = \frac{1}{720}$, when $z = 1$.

[illegible] $= 1$, $E = -4$, $F = 10$, and A, B, C, G &c. each $= 0$.

7. Required the ſum of the ſeries?

$$\frac{15 \cdot 18}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{18 \cdot 21}{2 \cdot 3 \cdot 4 \cdot 5} + \frac{21 \cdot 24}{3 \cdot 4 \cdot 5 \cdot 6} + \text{\&c.}$$

Here $T = \frac{9 \cdot \overline{z+4} \cdot \overline{z+5}}{z \cdot \overline{z+1} \cdot \overline{z+2} \cdot \overline{z+3}} = \frac{9}{z \cdot \overline{z+1}} + \frac{36}{z \cdot \overline{z+1} \cdot \overline{z+2}}$ [illegible] $\frac{18}{z \cdot \overline{z+1} \cdot \overline{z+2} \cdot \overline{z+3}}$; the values of z being 1, 2, 3 &c.

and $S = \frac{9}{z} + \frac{18}{z \cdot \overline{z+1}} + \frac{6}{z \cdot \overline{z+1} \cdot \overline{z+2}} =$

$\frac{9z^2 + 45z + 60}{z \cdot \overline{z+1} \cdot \overline{z+2}} = 19$ when z is taken equal 1.

ARTICLE XIV.

Tables of Theorems, for the calculation of Fluents, from Mr. Landen's *Memoirs, communicated by* Mr. William Burdon.

TABLE III.

(Continued from page 66.)

THEOREM XXI.

The *whole* fluent of $\frac{\dot{y}}{(a^2 - y^2)^{\frac{1}{2}}}$ is $= \frac{2L}{\sqrt{a}}$.

L THEOREM

THEOREM XXII.

$$\dot{F}=\frac{y^{\frac{1}{2}}\,\dot{y}}{(a^2-y^2)^{\frac{1}{2}}}.$$

$$F=K+\sqrt{2}\times(2E''-2e'e''-ae)-(ay-y^2)^{\frac{1}{2}}\times\left(\frac{a-y}{a+y}\right.$$

$$=K+\sqrt{2}\times(DP-AD)-(ay-y^2)^{\frac{1}{2}}\times\left(\frac{a-y}{a+y}\right)$$

$$x=a\times\left(\frac{a-y}{a+y}\right)^{\frac{1}{2}}.$$

THEOREM XXIII.

The fluent of $\frac{y^{\frac{1}{2}}\dot{y}}{(a^2-y^2)^{\frac{1}{2}}}$, generated whilst y from o becomes equal to $\frac{a}{\sqrt{2}}$, is $=\frac{L}{\sqrt{2}}$.

THEOREM XXIV.

The *whole* fluent of $\frac{y^{\frac{1}{2}}\dot{y}}{(a^2-y^2)^{\frac{1}{2}}}$ is $=L\sqrt{2}$.

THEOREM XXV.

$$\dot{F}=\frac{\dot{y}}{(y^2-a^2)^{\frac{1}{2}}}.$$

$$F=K+\frac{2^{\frac{3}{2}}}{\sqrt{a}}\times(ae+2e'e''-2E''+\tfrac{1}{2}DP)$$

$$=K+\frac{2^{\frac{3}{2}}}{\sqrt{a}}\times(AD-\tfrac{1}{2}DP).$$

$$x=y-\sqrt{y^2-a^2}.$$

THEOREM

THEOREM XXVI.

The fluent of $\frac{\dot{y}}{(y^2-a^2)^{\frac{1}{2}}}$, generated whilst y from a becomes equal to $a\sqrt{2}$, is $= \sqrt{2} \times (\sqrt{a} - \frac{L}{\sqrt{a}})$.

Note. The *whole* fluent is infinite.

THEOREM XXVII.

$$\dot{F} = \frac{y^{\frac{1}{2}}\dot{y}}{(y^2-a^2)^{\frac{1}{2}}}.$$

$$F = K + de - 2\,e'e'' + \frac{(y^2-a^2)^{\frac{1}{2}}*}{\sqrt{y}}$$

$$= K + DP - AD - L + \frac{(y^2-a^2)^{\frac{1}{2}}.*}{\sqrt{y}}$$

* A 3d proportional to DP, CP.

$$x = \frac{a\sqrt{y^2-a^2}}{y}.$$

THEOREM XXVIII.

The fluent of $\frac{y^{\frac{1}{2}}\dot{y}}{(y^2-a^2)^{\frac{1}{2}}}$, generated whilst y from a becomes equal to $a\sqrt{\frac{2^{\frac{1}{2}}+1}{2}}$, is $= \frac{1}{2} \times (a - L)$.

Note. The *whole* fluent is infinite.

THEOREM XXIX.

$$\dot{F} = \frac{\dot{y}}{(a^2+y^2)^{\frac{1}{2}}}.$$

$$F = K + \frac{2}{\sqrt{a}} \times (ae + 2e'e'' - 2E'' + DP.)$$

$$=K+\frac{2}{\sqrt{a}}\times AD.$$

$$x=\frac{a^2}{(a^2+y^2)^{\frac{1}{2}}}.$$

THEOREM XXX.

The fluent of $\frac{\dot{y}}{(a^2+y^2)^{\frac{1}{4}}}$, generated whilſt

becomes=to $\sqrt{2+\sqrt{2}}\times a$, is $=\sqrt{a}\times(\sqrt{2+1}$

Note, the *whole* fluent is infinite.

THEOREM XXXI.

$$\dot{F}=\frac{y^{\frac{1}{2}}\dot{y}}{(a^2+y^2)^{\frac{3}{4}}}.$$

$$F=K+de-2e'e''+\frac{y^{\frac{3}{2}}}{(a^2+y^2)^{\frac{1}{4}}}*$$

$$=K+DP-AD-L+\frac{y^{\frac{3}{2}}}{(a^2+y^2)^{\frac{1}{4}}}*.$$

* A
porti
DP

$$x=\frac{ay}{(a^2+y^2)^{\frac{1}{2}}}.$$

THEOREM XXXII.

The fluent of $\frac{y^{\frac{1}{2}}\dot{y}}{(a^2+y^2)^{\frac{1}{4}}}$, generated whilſt

becomes equal to $a\times\left(\frac{1}{\sqrt{2}}-\frac{1}{2}\right)^{\frac{1}{2}}$, is $=\frac{1}{2}\times$

Note, the *whole* fluent is infinite.

THE

THEOREM XXXIII.

$$\dot{F} = \frac{\dot{y}}{(a^2 - y^2)^{\frac{3}{4}}}.$$

$$F = K + \frac{4}{a^{\frac{3}{2}}} \times (ae + e'e'' - E'')$$

$$= K + \frac{2}{a^{\frac{3}{2}}} \times (ae + AD - DP).$$

$$x = (a^2 - y^2)^{\frac{1}{2}}.$$

THEOREM XXXIV.

The fluent of $\frac{\dot{y}}{(a^2 - y^2)^{\frac{3}{4}}}$, generated whilst y from o becomes equal to $\sqrt{2 - \sqrt{2}} \times a$, is $= \frac{M}{a^{\frac{3}{2}}}$.

THEOREM XXXV.

The *whole* fluent of $\frac{\dot{y}}{(a^2 - y^2)^{\frac{3}{4}}}$ is $= \frac{2M}{a^{\frac{3}{2}}}$.

THEOREM XXXVI.

$$\dot{F} = \frac{y^{-\frac{1}{2}}\dot{y}}{(a^2 - y^2)^{\frac{1}{4}}}.$$

$$F = K + \frac{2^{\frac{5}{2}}}{a^2} \times (ae + e'e'' - E'')$$

$$= K + \frac{2^{\frac{3}{2}}}{a^2} \times (ae + AD - DP).$$

$$x = a \times \left(\frac{a - y}{a + y}\right)^{\frac{1}{2}}.$$

THEOREM XXXVII.

The fluent of $\frac{y^{-\frac{1}{2}}\dot{y}}{(a^2-y^2)^{\frac{1}{2}}}$, generated whilst y from o becomes equal to $\frac{a}{\sqrt{2}}$, is $=\frac{2^{\frac{1}{2}}}{a^2}\times M$.

THEOREM XXXVIII.

The *whole* fluent of $\frac{y^{-\frac{1}{2}}\dot{y}}{(a^2-y^2)^{\frac{1}{2}}}$ is $=\frac{2^{\frac{3}{2}}}{a^2}\times M$.

THEOREM XXXIX.

$$\dot{F}=\frac{\dot{y}}{(y^2-a^2)^{\frac{1}{2}}}.$$

$$F=K+\frac{2^{\frac{3}{2}}}{a^{\frac{3}{2}}}\times(ae+e'e''-E'')$$

$$=K+\frac{2^{\frac{3}{2}}}{a^{\frac{3}{2}}}\times(ae+AD-DP).$$

$$x=y-\sqrt{y^2-a^2}.$$

THEOREM XL.

The fluent of $\frac{\dot{y}}{(y^2-a^2)^{\frac{1}{2}}}$, generated whilst y from a becomes equal to $a\sqrt{2}$, is $=\frac{\sqrt{2}}{a^{\frac{5}{2}}}\times M$.

THEOREM XLI.

The *whole* fluent of $\frac{\dot{y}}{(y^2-a^2)^{\frac{1}{2}}}$, is $=\overline{\frac{2}{a}}\Big|^{\frac{3}{2}}\times M$.

ARTICLE

ARTICLE XV.

NEW TABLES for finding the Contents of CASKS, by Mr. JOHN LOWRY, Author of a Treatise on Gauging, *to be published by Subscription.*

(Continued from p. 49.)

Quotient of the Head divided by the Bung.	Third Variety, or the two equal Frustums of a Paraboloid.		Fourth variety, or the two equal Frustums of a Cone.	
	A. G^s.	W. G^s.	A. G^s.	W. G^s.
·50	·0017407	·0021250	·0016247	·0019833
·51	·0017548	·0021421	·0016433	·0020061
·52	·0017692	·0021596	·0016622	·0020291
·53	·0017838	·0021775	·0016812	·0020523
·54	·0017987	·0021957	·0017047	·0020758
·55	·0018139	·0022142	·0017198	·0020995
·56	·0018293	·0022331	·0017394	·0021234
·57	·0018451	·0022523	.0017592	·0021475
·58	·0018611	·0022718	·0017792	·0091719
·59	·0018774	·0022917	·0017993	·0021965
·60	·0018939	·0023119	·0018196	·0022213
·61	·0019108	·0023325	·0018402	·0022464
·62	·0019279	·0023534	·0018609	·0022716
·63	·0019453	·0023747	·0018817	·0022971
·64	·0019630	·0023963	·0019028	·0023228
·65	·0019809	·0024182	·0019241	·0023488
·66	·0019992	·0024388	·0019455	·0023750
·67	·0020177	·0024631	·0019672	·0024014
·68	·0020365	·0024860	·0019891	·0024280
·69	·0020556	·0025093	·0020110	·0024549
·70	·0020749	·0025329	·0020332	·0024820
·71	·0020946	·0025569	·0020555	·0025093
·72	·0021145	·0025812	·0020779	·0025368
·73	·0021347	·0026059	·0021007	·0025646

Quotient of the Head divided by the Bung.	Third Variety, or the two equal Frustums of a Paraboliod.		Fourth Variety, or the two equal Frustums of Cone.	
	A. G^s.	W. G^s.	A. G^s.	W. G^s.
·74	·0021552	·0026389	·0021236	·0025926
·75	·0021759	·0026562	·0021467	·0026208
·76	·0021969	·0026818	·0021700	·0026493
·77	·0022183	·0027079	·0021935	·0026779
·78	·0022398	·0027342	·0022172	·0027068
·79	·0022617	·0027609	·0022411	·0027360
·80	·0022838	·0027879	·0022653	·0027653
·81	·0023063	·0028153	·0022895	·0027949
·82	·0023290	·0028430	·0023139	·0028247
·83	·0023519	·0028711	·0023385	·0028547
·84	·0023752	·0028994	·0023633	·0028850
·85	·0023987	·0029282	·0023883	·0029155
·86	·0024225	·0029572	·0024134	·0029462
·87	·0024466	·0029807	·0024388	·0029771
·88	·0024710	·0020164	·0024634	·0030083
·89	·0024956	·0030465	·0024900	·0030397
·90	·0025206	·0030769	·0025159	·0030713
·91	·0025459	·0031077	·0025420	·0031034
·92	·0025713	·0031388	·0025683	·0031352
·93	·0025970	·0031703	·0025947	·0031675
·94	·0026231	·0032020	·0026214	·0032001
·95	·0026494	·0032342	·0026482	·0032328
·96	·0026760	·0032666	·0026752	·0032658
·97	·0027029	·0032995	·0027024	·0032990
·98	·0027300	·0033326	·0027300	·0033324
·99	·0027575	·0033661	·0027574	·0033661
1·00	·0027851	·0033999	·0027851	·0033999

To

To the Editor.

SIR,

There is no part of Practical Mathematics where the opinions of authors have been so very different as in the business of *Cask Gauging*; for, among the numerous writers on the subject, it is almost impossible to find any two that agree with each other, some differing in the form of the casks, and others in the methods of finding their contents: with most authors, however, it has been usual to divide casks into four varieties or forms, *viz.*

1. The middle frustum of a spheroid.
2. The middle frustum of a parabolic spindle.
3. The two equal frustums of a paraboloid.
4. The two equal frustums of a cone.

Now, Sir, as you have already exhibited to your readers, tables for each of the above varieties, by which the contents may be found pretty near the truth, and that in much less time than by any other tables for the same purpose; but as these forms or varieties are only imaginary, and it is very probable that there never was a cask that agreed with any one of the four varieties.

Therefore I wish to recommend to my brother officers the following *General Rule*, taken from Dr. Hutton's *excellent* Mathematical and Philosophical Dictionary, vol. 1. page 258, the accuracy of which (the Doctor observes) "has been verified and proved by filling the cask with a true gallon measure."

General Rule.

Add into one sum,

39 times the square of the bung diameter,
25 times the square of the head diameter, and
26 times the product of those diameters;

Multiply the sum by the length of the cask, and the product by the number ·00034; then this last product

product divided by 9 will give the *wine gallons*, and divided by 11 will give the *ale gallons*.

This rule (though much readier in practice than any other general rule I have yet met with) requires a great many figures in the operation, and cannot easily be performed by the sliding rule; I have therefore calculated the following table of multipliers and gauge points, by which the contents of any cask may be found by one single operation of the sliding rule:

A GENERAL TABLE

For finding the Content of any Cask, either by the Pen or Sliding Rule.

Quoti. of Head div. by Bung.	Multipliers for A. G'.	Gauge points for A. G'.	Multipliers for W. G'.	Gauge points for W. G'.
·50	·0018026	23·55	·0022006	21·31
·51	·0018184	23·45	·0022199	21·22
·52	·0018345	23·35	·0022395	21·13
·53	·0018507	23·24	·0022592	21·04
·54	·0018670	23·14	·0022791	20·95
·55	·0018835	23·04	·0022992	20·85
·56	·0019001	22·94	·0023196	20·76
·57	·0019170	22·84	·0023401	20·67
·58	·0019337	22·74	·0023608	20·57
·59	·0019509	22·64	·0023817	20·48
·60	·0019681	22·54	·0024027	20·39
·61	·0019855	22·45	·0024239	20·30
·62	·0020031	22·35	·0024454	20·21
·63	·0020207	22·25	·0024670	20·13
·64	·0020386	22·15	·0024888	20·04
·65	·0020567	22·05	·0025108	19·95
·66	·0020932	21·86	·0025330	19·87
·67	·0020750	21·95	·0025553	19·78

Quotient of the Head divided by the Bung.	Multipliers for A. G^s.	Gauge points for A. G^s.	Multipliers for W. G^s.	Gauge points for W. G^s.
·68	·0021116	21·76	·0025780	19·69
·69	·0021303	21·66	·0026007	19·60
·70	·0021491	21·56	·0026237	19·51
·71	·0021680	21·47	·0026468	19·43
·72	·0021871	21·37	·0026701	19·35
·73	·0022064	21·28	·0026936	19·26
·74	·0022258	21·18	·0027173	19·18
·75	·0022454	21·09	·0027412	19·09
·76	·0022651	21·01	·0027653	19·01
·77	·0022850	20·92	·0027896	18·93
·78	·0023050	20·83	·0028141	18·85
·79	·0023253	20·74	·0028387	18·77
·80	·0023457	20·65	·0028635	18·69
·81	·0023660	20·56	·0028885	18·61
·82	·0023867	20·47	·0029137	18·53
·83	·0024075	20·38	·0029392	18·44
·84	·0024285	20·29	·0029647	18·36
·85	·0024496	20·20	·0029905	18·28
·86	·0024710	20·12	·0030165	18·20
·87	·0024923	20·03	·0030426	18·13
·88	·0025140	19·95	·0030690	18·05
·89	·0025357	19·86	·0030955	17·97
·90	·0025576	19·77	·0031223	17·89
·91	·0025796	19·68	·0031493	17·81
·92	·0026019	19·60	·0031763	17·74
·93	·0026242	19·52	·0032036	17·67
·94	·0026467	19·44	·0032311	17·59
·95	·0026693	19·36	·0032588	17·51
·96	·0026922	19·27	·0032867	17·44
·97	·0027152	19·18	·0033147	17·37
·98	·0027384	19·10	·0033428	17·30
·99	·0027616	19·02	·0033713	17·22
1·00	·0027851	18·95	·0034000	17·15

To

To find the content of any cask by the above table.

By the Pen.

Use the general rule on page 49, art. 4.

By the Sliding Rule.

To the gauge point on D (found opposite to the quotient arising from dividing the head diameter by the bung,) set the length of the cask on C, then against the bung diameter on D you have the content on C.

Example.

Required the content of a cask whose length is 40, bung diameter 32, and head diameter 24 inches.

By the Pen.

First. $24 \div 32 = \cdot 75$, against which in the table is ·0022454 for ale gallons, and ·0027412 for wine gallons.

$$\left.\begin{array}{l}\text{Hence } \cdot 0022454 \times \overline{32}|^2 \times 40 = 91 \cdot 97 \text{ ale} \\ \text{and } \cdot 0027412 \times \overline{32}|^2 \times 40 = 112 \cdot 27 \text{ wine}\end{array}\right\} \begin{array}{l}\text{gallons,} \\ \text{the con-} \\ \text{tent re-} \\ \text{quired.}\end{array}$$

By the Sliding Rule.

The gauge points opposite ·75 are 21·09 for ale gallons, and 19·09 for wine gallons, hence by the rule,

$$\text{as } \left.\begin{array}{r}21 \cdot 09 \\ 19 \cdot 09\end{array}\right\} \text{on D} : 40 \text{ on C} :: 32 \text{ on D} : \left\{\begin{array}{l}92 \text{ ale galls. on C,} \\ 112 \cdot 3 \text{ wine g. on C.}\end{array}\right.$$

Having shewn the use of the above table, I shall conclude with these two observations.

1. These gauge points may be very distinctly fixed on a two foot rule, in the following manner, viz. at 21·31 on the line D place ·50, and at 21·22, 21·13, 21·04 &c place 1, 2, 3, 4, 5, 6, 7, 8, 9, ·60, 1, 2, 3 &c. to 1·00 which last falls in the circular guage point for wine gallons; and in the same manner may the gauge

gauge points for ale gallons be fixed either above or below the other.

2. The above operation by the *Sliding Rule* being ſo very eaſy and accurate, I ſhould hope to ſee it generally practiſed by the *Exciſe*, were not the generality of my brother officers ſo extremely prejudiced in favour of their old erroneous methods that it is difficult to perſuade them to try any other. It is much to be wiſhed that the Honourable Board of Exciſe would adopt ſome general rules for gauging and ullaging caſks, the want of which, I am certain, is often highly injurious to the revenue, as well as to the fair trader.

JOHN LOWRY.

ARTICLE XVI.

Anſwers to Queſtions propoſed in the Proſpectus.

I. QUESTION 1. *from Mr.* Lawſon's *Diſſertation on the Geometrical Analyſis of the Ancients.*

IF a right line AB be biſected in E, and the points C and D taken therein ſuch that AC : CB :: AD : DB; then I ſay the rectangle DCE = the rectangle ACB.

The converſe of this is alſo true, which is this.

If a right line AB be biſected in E, and the points C and D taken therein ſuch that DCE = ACB; then I ſay AC : CB :: AD : DB. Required the Demonſtration?

Anſwered by Peletarius *.

ANALYSIS.

Suppoſe the theorem true. Let CF (Fig. 54, 55, Pl. 3.) be made perpendicular and equal to BC; join

* The Editor would be glad if PELETARIUS would favour him with his Addreſs.

BF and draw EG parallel to BF meeting CF in G; join DF, DG, AF and AG; produce AG till it meets BF in H, and draw HK parallel to AB meeting CF in K.

Then since rect. $DCE =$ rect. ACB,
and $GC = CE$,
we have rect. $DCE = 2 \triangle DCG$,
and rect. $ACB = 2 \triangle ACF$;
therefore $\triangle DCG = \triangle ACF$,
from which take the common $\triangle ACG$,
and there remains $\triangle DAG = \triangle FAG$,
therefore AG, DF are parallel,
wherefore, it will be $HF : FB :: AD : DB$.
Again, since $AC : HK :: AG : GH$;
and, because BF is parallel to EG,
it will be $AG : GH :: AE : EB$,
hence by equality, $AC : HK :: AE : EB$;
but, because $AE = EB$, we shall have $AC = HK$;
wherefore $AC : CB :: HK : CB :: HF : FB :: AD : DB$.

Q. Q. V.

SYNTHESIS.

Since $AC : CB :: AD : DB$,
and $AD : DB :: HF : FB :: HK : CB$,
we have $AC : CB :: HK : CB$,
therefore $AC = HK$ and $AG = GH$;
but $AE = EB$,
wherefore $AG : GH :: AE : EB$,
therefore EG, BF are parallel,
and therefore $GC : CE :: FC : CB$,
but $FC = CB$, wherefore $GC = CE$.
Again, since AG, DF are parallel
we have $\triangle ADG = \triangle AFG$,
add the common $\triangle ACG$ to each,
and $\triangle DCG = \triangle ACF$;
wherefore $2 \triangle DCG = 2 \triangle ACF$
but, rect. DCG or $DCE = 2 \triangle DCG$

and

and rect. ACF or ACB $=$ 2 $\triangle$ ACF;
therefore rect. ACB $=$ rect. DCE.

Q. E. D.

Conversely.

ANALYSIS.

Suppose it true. Let CF be made perpendicular and equal to BC; join AF, BF, DF and draw AG parallel to DF meeting CF, BF, in G, H respectively; also draw HK parallel to AB meeting CF in K; join DG, EG.

Since AC : CB :: AD : DB,
and AD : DB :: HF : FB :: HK : CB;
therefore by equality, AC : CB :: HK : CB,
but the consequents CB, CB are equal,
therefore the antecedents AC, HK must be equal;
wherefore AG $=$ GH, but AE $=$ EB,
wherefore AG : GH :: AE : EB;
therefore EG, BF are parallel;
hence GC : CE :: FC : CB,
but FC $=$ CB; and therefore GC $=$ CE.
Again, since AG, DF are parallel,
we have $\triangle$ ADG $=$ $\triangle$ AFG,
add the common $\triangle$ ACG to each,
and so shall $\triangle$ DCG $=$ $\triangle$ ACF,
wherefore 2 $\triangle$ DCG $=$ 2 $\triangle$ ACF;
but rect. DCG or DCE $=$ 2 $\triangle$ DCG,
and rect. ACF or ACB $=$ 2 $\triangle$ ACF,
and therefore rect. DCE $=$ rect. ACB.

Q. Q. V.

SYNTHESIS.

Since rect. DCE $=$ rect. ACB,
and GC $=$ CE,
we shall have rect. DCE $=$ 2 $\triangle$ DCG,
and also rect. ACB $=$ 2 $\triangle$ ACE;

therefore $\triangle DCG = \triangle ACF$;
take the common $\triangle ACG$ from each,
and there will remain $\triangle DAG = \triangle FAG$,
therefore AG, DF are parallel;
wherefore $HF : FB :: AD : DB$.
Again, since $AC : HK :: AG : GH$,
and BF is parallel to EG;
we shall have $AG : GH :: AE : EB$;
and by equality $AC : HK :: AE : EB$,
but $AE = EB$; and therefore $AC = HK$;
wherefore $AC : CB :: HK : CB :: HF : FB :: AD : DB$.

Q. E. D.

The same answered by Mr. Collin Campbell.

Since $AC : CB :: AD : DB$, (Fig. 54.)
invertendo $CB : AC :: DB : AD$,
dividendo $2CE : AC :: AB : AD$,
hence $CE : AC :: BE : AD$;
permutando $CE : BE :: AC : AD$,
invertendo $BE : CE :: AD : AC$,
componendo $CB : CE :: DC : AC$,
wherefore rect. $DCE =$ rect. ACB.

Q. E. D.

Conversely.

Because rect. $DCE =$ rect. ACB,
it will be $DC : AC :: CB : CE$;
and dividendo $AD : AC :: EB : CE$,
permutando $AD : EB :: AC : CE$;
or, $AD : AB :: AC : 2CE$;
componendo $AD : BD :: AC : CB$,
that is $AC : CB :: AD : DB$.

Q. E. D.

The same demonstrated by Mr. John Lowry.

By hypothesis $AC : CB :: AD : DB$,

and

and, by perm. compos. &c. DC : AC :: 2CB$\pm$DC : CB ;
but AB is bisected in E ; and therefore CB=2C$\pm$AC ;
hence DC : AC :: 2CB$\pm$DC : 2CE$\pm$AC,
and by perm. compos. &c. DC : CB :: AC : CE
that is rect. DC·CE=rect. AC·CB.

Q. E. D.

Conversely.

By hypothesis rect. DC·CE=rect. AC·CB,
therefore DC : CB :: AC : CE,
and by perm. compos. &c. DC : AC :: 2CB$\pm$DC : 2CE$\pm$AC,
but AB is bisected in E ; therefore 2CE$\pm$AC=CB ;
wherefore DC : AC :: 2CB$\pm$DC : CB,
therefore by division AD : AC :: DB : CB,
hence AC : CB :: AD : DB.

Q. E. D.

II. QUESTION 2. *from the same.*

If in AB the diameter of a circle two points C and D be assumed such that AC:CB::AD:DB, and from D an indefinite perpendicular to the same diameter as LD be erected, and through C any line be drawn to cut the same in E and the circle in F and G ; I say FC : CG :: FE: EG:

The Converse of this is also true, which is this :

If any right line as LD be drawn perpendicular to the diameter AB of any circle and meet the same in D, and if from a point in the same diameter as C, any line be drawn to meet the same perpendicular in E and the circle in F and G, so that FC : CG :: FE: EG, I say that AC : CB :: AD: DB. Required the Demonstration ?

Answered by Peletarius.

ANALYSIS.

Suppose it true. Let H (Fig. 56, 57, Plate 3.) be the centre of the circle, bisect FG in K and join HK.

Then, since FC : CG :: FE : EG:
and FG is bisected in K,
by the last Question we shall have rect. ECK = rect. FCG;
again, because FG is bisected in K,
and H is the centre of the circle;
the angle CKH will be a right angle,
but, the angle CDE is a right angle;
therefore the points E, D, H, K, are in the circum. of a circle;
wherefore rect. FCK = rect. DCH,
and rect. FCG = rect. ACB;
consequently rect. DCH = rect. ACB.
but, AB is bisected in H,
and in AB two points C, D, are found,
such that the rect. DCH = rect. ACB; and
theref. by converse of last Qu. AC : CB : : AD : DB.

Q. Q. V.

SYNTHESIS.

Since AC : CB : : AD : DB,
and AB is bisected in H;
therefore by the last Qu. rect. DCH = rect. ACB — rect. FCG.
again, because H is the centre of the circle,
and FG is bisected in K;
the angle CKH will be a right angle,
but the angle CDE is a right angle;
therefore the points E, D, H, K, are in the circum. of a circle;
wherefore rect. ECK = rect. DCH = rect. FCG;
but, FG is bisected in K,
and in FG two points C, E, are found,
such that rect. ECK = rect. FCG
wherefore by converse of last Qu. FC : CG :: FE : EG.

Q. E. D.

Conversely.

Converſely.

ANALYSIS.

Suppoſe it true. Let H be the centre of the circle, biſect FG in K and join HK.

Becauſe AC : CB :: AD : DB,
and AB is biſected in H;
therefore by the laſt Qu. rect. DCH = rect. ACB = rect. FCG.
again, becauſe H is the centre of the circle,
and FG is biſected in K;
the angle CKH will be a right angle,
but the angle CDE is a right angle;
therefore the points E, D, H, K are in the circum. of a circle;
wherefore rect. ECK = rect. DCH = rect. FCG;
but FG is biſected in K,
and in FG two points C, E, are found,
ſuch that, rect. ECK = rect. FCG;
therefore by converſe of laſt Qu. FC : CG :: FE : EG.

Q. Q. V.

SYNTHESIS.

Since FC : CG :: FE : EG,
and FG is biſected in K;
therefore by the laſt Qu. rect. ECK = rect. FCG = rect. ACB;
but becauſe H is the centre of the circle,
and FG is biſected in K;
the angle CKH will be a right angle,
but the angle CDE is a right angle;
therefore the points E, D, H, K are in the circum. of a circle;
wherefore rect. DCH = rect. ECK = rect. ACB;
but AB is biſected in H,
and in AB, two points C, D, are found
ſuch that, rect. DCH = rect. ACB;
therefore by converſe of the laſt Qu. AC : CB :: AD : DB.

Q. E. D.

The ſame anſwered by Meſſ. Campbell *and* Lowry.

Biſect the diameter AB in H, and draw HK perpendicular to GE.

Then

Then by ſim. triangles HC : KC :: CE : ED,
therefore rect. HCD = rect. KCE;
but by laſt Qu. and Eu. III. 35, rect. HCD = rect. ACB = rect. GCF,
Hence rect. KCE = rect. GCF;
and by converſe of laſt Qu. FC : CG :: FE : EG.
Q. E. D.

Converſely.

Since FC : CG :: FE : EG,
by the laſt Qu. and Eu. III. 35, rect. KCE = rect. GCF = rect. ACB;
alſo by ſim. triangles, DC : EC :: CK : CH,
therefore rect. KCE = rect. DCH;
wherefore rect. ACB = rect. DCH;
therefore by the converſe of the laſt Qu. AC : CB :: AD : DB.
Q. E. D.

III. QUESTION 3. *from* Emerſon's *Fluxions*, *Page* 66.

Required the Fluent of

$$(ax^2+bxy+cy^2)\times\dot{x}+(dx^2+fxy+gy^2)\times\dot{y}=0.$$

Anſwered by Mr. John Surtees.

Aſſume $(x+\alpha y)^{\pi}\cdot(x+\beta y)^{\tau}\cdot(x+\gamma y)^{s}=A$, a given quantity; then, by logarithms,

$$\pi L\cdot(x+\alpha y)+\tau L\cdot(x+\beta y)+s L\cdot(\dot{x}+\gamma y)=L\cdot A,$$

in fluxions $\pi\dfrac{\dot{x}+\alpha\dot{y}}{x+\alpha y}+\tau\dfrac{\dot{x}+\beta\dot{y}}{\dot{x}+\beta y}+s\dfrac{\dot{x}+\gamma y}{x+\gamma y}=0.$

hence by reduction we have,

$$\left.\begin{array}{r}(\pi+\tau+s)\cdot x^2\dot{x}\\ +(\pi\cdot\overline{\beta+\gamma}+\tau\cdot\overline{\alpha+\gamma}+s\cdot\overline{\beta+\alpha})\cdot xy\dot{x}\\ +(\pi\alpha\cdot\overline{\beta+\gamma}+\tau\beta\cdot\overline{\alpha+\gamma}+s\gamma\cdot\overline{\beta+\alpha})\cdot xy\dot{y}\\ +(\pi\alpha+\tau\beta+s\gamma)\cdot x^2\dot{y}\\ +(\beta\pi\gamma+\tau\alpha\gamma+s\alpha\beta)\cdot y^2\dot{x}\\ +(\alpha\beta\gamma\cdot(\pi+\tau+s)\cdot y^2\dot{y}\end{array}\right\}=0.$$

Whence

Whence by comparing the terms with thofe of the given equation we obtain, thefe fix equations, viz.

$$a = \pi + \tau + s,$$
$$b = \pi \cdot (\beta + \gamma) + \tau \cdot (\alpha + \gamma) + s \cdot (\alpha + \beta),$$
$$c = \pi\beta\gamma + \tau\alpha\gamma + s\alpha\beta,$$
$$d = \pi\alpha + \tau\beta + s\gamma,$$
$$f = \pi\alpha \cdot (\beta + \gamma) + \tau\beta \cdot (\alpha + \gamma) + s\gamma \cdot (\beta + \alpha),$$
$$g = \alpha\beta\gamma \cdot (\pi + \tau + s).$$

From hence the value of π, τ, s, α, β, γ, may be found in terms of a, b, c, d, f, g, and then by fubftitution, the affumed expreffion may be exhibited with known co-efficients and exponents, which will define the fluent of the propofed fluxion.

Exactly in this manner the fluent was found by Mr. Collin Campbell, *of Kendal.*

IV. QUESTION 4. *from* Stewart's *General Theorems.*

Let there be any number of given points; a point may be found, fuch, that if from all the given points there be drawn right lines to the point found, and from all the given points and the point found there be drawn right lines to any point, the fum of the fquares of the lines drawn from the given points will be equal to the fum of the fquares of the lines drawn from the given points to the point found, together with the multiple, by the number of the given points of the fquare of the line drawn from the point found. Required the Demonftration?

N. B. This is the 9th prop. of the above book, and is the firft that is left undemonftrated by the author.

Anfwered

Answered by Dr. Small, from the Transactions of the Royal Society of Edinburgh, vol. 11.

Dr. Small delivers this question, and its investigation in the following manner:

Let there be any number, m, of given points A, B, C, &c. a point X may be found such that if from A, B, C, &c. there be drawn straight lines to any point D, and to the point X found, and if DX, be joined,

$$AD^2+BD^2+CD^2 \text{ \&c.} = AX^2+BX^2+CX^2 \text{ \&c.} + m\,DX^2.$$

Let $m=3$. (Fig. 58, Plate 3.)

Suppose the point X found, join DX, from the given points A, B, C, draw AE, BF, CG perpendicular to DX, and join AX, BX, CX.

Since $AD^2+BD^2+CD^2=AX^2+BX^2+CX^2+3DX^2$, and
$AD^2=AX^2+DX^2-2DX\cdot XE$, and
$BD^2=BX^2+DX^2+2DX.XF$, and
$CD^2=CX^2+DX^2+2DX.XG$, the point X in the line DX must be so taken, that the part EX, intercepted between it and AE the perpendicular, from the point A, be equal to FX and GX, the sum of the parts intercepted between it and the perpendiculars BF and CG, from B and C; and the parts FX, GX must be in the opposite direction to EX.

This will be effected by the following construction: join AB, and bisect it in H; and join CH, and divide it in X, so that $CX=2HX$; X will be the point required.

From H draw DX the perpendicular HK.

Since $AH=BH$, we shall have $EK=FK$;
and since $CX=2HX$, we shall also have $GX=2KX$.

Therefore since $FX=FK-KX$, and
$GX=2KX$
$FX+GX=FK+KX=EK+KX=EX$, and
$-2DX\cdot XE+2DX\cdot XF+2DX\cdot XG=0.$

The

The point X thus found is the centre of gravity of the three points A, B, C.

The ſecond and fourth of Dr. *Stewart's* Theorems are particular caſes of this propoſition, and are eaſily derived from it.

Remark. Dr. Small's *demonſtration of this propoſition, as well as his other demonſtrations in the* Edinburgh Tranſactions, *are very ingenious, but they certainly lack much of that geometrical purity, and chaſtity of expreſſion, which are to be found in the demonſtrations of the firſt five theorems left by Dr.* Stewart, (which is given in Article II. of this work,) *as patterns to be followed in the demonſtration of the reſt.*

The Editor invites Geometricians to exert themſelves, and to dive into *Dr. Stewart's* ſcientific fiſhery, for the rich pearls which he promiſes them; for the Doctor ſays, " if any give themſelves the trouble to explain ſome of theſe theorems, they will find their time and pains ſufficiently rewarded, by the great number of new and curious propoſitions which they would infallibly diſcover in doing it. "

The ſame anſwered by Mr. Lowry.

Let A, B, C, D, E, F, &c. (Fig. 36, Plate 2.) be the given points, join any two of them as A, B, with the right line AB which biſect in I; join the point I and any other of the given points as C with the right line IC; cut IC in P, ſo that PC=2PI; join the point P and any other of the given points as D with the right line PD which divide in H ſo that DH=3PH; join the point H and any other of the given points as E with the right line HE, divide HE in G ſo that EG=4GH; join the point G and any other of the given points as F with the right

right line GF, divide GF in K so that $FK=5GK$; join the point K and any other of the given points, &c. then will I be the point required for the two given points A, B; P for the three given points A, B, C; H for the four given points A, B, C, D; G for the five given points A, B, C, D, E; K for the six given points, A, B, C, D, E, F, &c. that is supposing Q any other point, and right lines be drawn from it to all the given points, and the point found, and right lines be drawn from the given points to the point found, then will

$$AQ^2+BQ^2=AI^2+BI^2+2IQ^2,$$
$$AQ^2+BQ^2+CQ^2=AP^2+BP^2+CP^2+3PQ^2,$$
$$AQ^2+BQ^2+CQ^2+DQ^2=AH^2+BH^2+CH^2+DH^2+4HQ^2,$$
&c. &c.

for two, three, four, &c. points.

DEMONSTRATION.

Case 1. For two given points A, B.

Because A B is bisected in I, we have by Prop. II. Art. II. $AQ^2+BQ^2=2AI^2+2IQ^2,$
$$=AI^2+BI^2+2IQ^2.$$
Q. E. D.

Case 2. For three given points A, B, C.

On IPC demit the perpendicular QL, then by Prop. II. Art. II. $AP^2+BP^2=2AI^2+2IP^2,$
and by *cons.* $PC=2IP$, or $PC^2=4IP^2$,
therefore $AP^2+BP^2+CP^2=2AI^2+6IP^2$.
Again $AQ^2+BQ^2=2AI^2+2IQ^2,$
Euclid I. 47. $=2AI^2+2IL^2+2LQ^2,$
also $QC^2=LQ^2+LC^2,$
and therefore $AQ^2+BQ^2+CQ^2=2AI^2+2IL^2+CL^2+3LQ^2$;
but $IL=IP+PL$ and $CL=2IP-PL$
thererefore $2IL^2+CL^2=6IP^2+3PL^2,$
and $3PQ^2=3QL^2+3PL^2,$

where-

wherefore $AQ^2+BQ^2+CQ^2=2AI^2+6IP^2+3PQ^2$;
but $AP^2+BP^2+CP^2=2AI^2+6IP^2$,
therefore $AQ^2+BQ^2+CQ^2=AP^2+BP^2+CP^2+3PQ^2$.
Q. E. D.

Case 3. For four given points A, B, C, D.

On PD, PC demit the perpendiculars QN, HS;
Then Prop. II. Art. II. $AH^2+BH^2=2AI^2+2IH^2$,
Eu. I. 47. $=2AI^2+2IS^2+2SH^2$;
and $CH^2=CS^2+SH^2$,
therefore $AH^2+BH^2+CH^2=2AI^2+2IS^2+CS^2+3SH^2$;
but $IS=IP+PS$ and $CS=2IP-PS$,
and $3PH^2=3PS^2+3SH^2$,
therefore $2IS^2+CS^2=6IP^2+3PS^2$,
and $AH^2+BH^2+CH^2=2AI^2+6IP^2+3PH^2$;
by *conf.* $DH=3PH$, or $DH^2=9PH^2$,
therefore $AH^2+BH^2+CH^2+DH^2=2AI^2+6IP^2+12PH^2$.
Again $AQ^2+BQ^2+CQ^2=2AI^2+6IP^2+3PQ^2$,
Euclid I. 47. $=2AI^2+6IP^2+3PN^2+3QN^2$,
and $DQ^2=QN^2+ND^2$, therefore
$AQ^2+BQ^2+CQ^2+DQ^2=2AI^2+6IP^2+3PN^2+ND^2+4QN^2$;
but $DH=3PH$,
therefore $PN=PH+HN$ and $ND=3PH-HN$;
hence $3PN^2+ND^2=12PH^2+4HN^2$,
but $4HQ^2=4NQ^2+4HN^2$,
wher. $AQ^2+BQ^2+CQ^2+DQ^2=2AI^2+6IP^2+12PH^2+4HQ^2$;
but $AH^2+BH^2+CH^2+DH^2=2AI^2+6IP^2+12PH^2$, therefore
$AQ^2+BQ^2+CQ^2+DQ^2=AH^2+BH^2+CH^2+DH^2+4HQ^2$

In the same manner may the proposition be proved for any number of points, but as the number increases, the demonstration becomes more prolix.

V. QUESTION 5. *by Mr.* Thomas Bournley.

Let there be a semicircle whose diameter is AB, and centre C; bisect CB in D, and draw DE per-

pendicular to AB, meeting the ſemicircle in E; join AE; if from any point F, in the ſemicircle, there be drawn FG perpendicular to AB, meeting AB in G and AE in H, the ſquare of DE will be greater than the rectangle FGH. Required the Demonſtration?

Anſwered by Pappus, junior.

Let EK, (Fig. 59, Plate 3.) a tangent to the ſemicircle at E, meet AB in K and FG in L: then by the nature of the circle CK, CB, CD, are proportionals;

but $CB = 2CD$; and therefore $CK = 2CB$;

wherefore $KB = BC = AC$; but $BD = DC$,

and, therefore $AD = DK$.

Again, $AD^2 :$ rect. $DAG :: DA : AG :: DE : GH :: DE^2 : DE \cdot GH$,

and alternately $AD^2 :: DE^2 ::$ rect. $DAG :$ rect. $DE \cdot GH$;

again rect. $AGK : DAG :: GK : AD$ or $DK :: LG : DE :: LGH : DE \cdot GH$,

alternately rect. $AGK : LGH :: DAG : DE \cdot GH$:

hence by equality, $AD^2 : DE^2 :: AG \cdot GK : LG \cdot GH$;

but AD^2 is greater than rect. AGK,

therefore DE^2 is greater than rect. LGH;

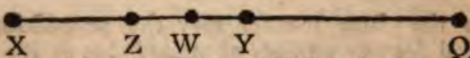
and therefore, becauſe GL is greater than GF,

DE^2 is greater than rect. FGH.

Q. E. D.

The ſame demonſtrated by Mr. Campbell.

LEMMA.



If any right line XQ be biſected in Y, and any how divided in the points Z, W, ſo that W be nearer to Y than Z; I ſay rect. XWQ is greater than rect. XZQ.

This being premiſed, Take the point F any where between AE; and through any point f in the arch AB make hfg perpendicular to AB, meeting AE produced in h, draw the lines FET, Efr; meeting AB

AB produced in T, r; also let EK, a tangent to the semicircle at E, meet AB produced in K.

Then, by right angled △'s, rect. $DC \cdot CK = CE^2$,

and $CK = CE^2 \div CD = 2CE$;

and therefore $AD = DK$.

Case 1. To prove that DE^2 is greater than rect. FGH.

By similar triangles, $AG : AD :: GH : DE$,

and $TG : TD :: GF : DE$;

and, by Simpson's Geo. IV. 11, rect. $AGT : ADT :: HGF : DE^2$;

but DT is evidently greater than DK,

and D is nearer the middle of AT than G;

therefore by the lemma rect. AGT is less than rect. ADT;

consequently rect. HGF is less than DE^2.

Case 2. To prove that DE^2 is greater than rect. fgh.

By similar triangles, $AD : Ag :: DE : gh$,

and $Dr : gr :: DE : gf$;

therefore rect. $ADr : Agr :: DE^2 : hgf$.

Now, Dr being less than DK, it is less therefore than AD,

but, D is nearer the middle of Ar than g;

therefore by the lemma rect. Agr is less than rect. ADr,

wherefore rect. fgh is less than DE^2.

Q. E. D.

VI. QUESTION 6. *from the London Magazine improved.*

Suppose a plane to touch the spheroidal figure of the earth, in a given latitude; it is required to find the angle contained by this plane and a tangent drawn to any given point of the earth?

Answered by nobody.

It is therefore re-proposed in the present Number.

VII. QUESTION 7. *from the British Oracle.*

Required the area of the common parabola by the method of increments?

Answered by Mr. Collin Campbell.

If an infinite number of double ordinates were drawn, so that the parabola might be divided into indefinitely small areas, it is manifest that these evanescent spaces, bounded by every two adjoining ordinates and the curve, may truly be considered as parallelograms, whose sum will express the area sought.

Put $x =$ any variable part of the abscissa,
$y =$ the corresponding ordinate,
and $r =$ the latus rectum; then by conics, $rx = y^2$, and $rx_1 = y_1^2$, or $rx + r\dot{x} = y^2 + 2y\dot{y} + \dot{y}^2$;

hence $r\dot{x} = 2y\dot{y} + \dot{y}^2 = 2y\dot{y}$, because $\dot{y}$ is evanescent;

wherefore $2y\dot{x}$, the area of an incremental parallelogram, is $= \frac{4y^2\dot{y}}{r} = \frac{4}{r} \times (y y_1 \dot{y} - y\dot{y}^2)$; and the integral thereof $= \frac{4}{r} \times \left(\frac{y y_1 \dot{y}}{3} - \frac{y y_1 \dot{y}}{2} \right)$

$= \frac{4}{r} \times \left(\frac{y^3}{3} - 0 \right) = \frac{4}{3r} y^3 = \frac{4}{3r} y \times rx = \frac{4}{3} xy$ the area, as required.

VIII. QUESTION 8. *being Prop. 1st of* Mac Laurin's *Geometria Organica.*

Circa duo puncta C & S in Plano quovis data tanquam Polos moveantur Anguli dati FCO & KSH; ducatur concursus crurum CF & SK per rectam AE in eodem Plano positione datam; atque reliquæ interea Crura CO & SH Concursu suo P describent Curvam

Curvam primi Genesis.—Required an English Translation and Solution ?

Answered from Mr. Mac Laurin's *Algebra.*

TRANSLATION.

Let the two points C and S be given, and the straight line AE in the same plane; Let the given angles FCO,KSH, revolve about the points C and S as poles, and let the intersection of the sides CF, SK be carried along the straight line AE; and the intersection of the sides CO, SH, will describe a curve of the first order.

Investigation.

Let the sides CF, SK, (Fig. 60, Plate 3.) intersect each other in Q, and the sides CO, SH in P; let PM and QN be perpendicular on CS; then draw PR, QU; PT, QL so that, $\angle$CUQ $=$CRP$=$FCG; and $\angle$, SLQ$=$STP$=$KSD; since the angle RCP makes two right ones with RCQ and QUC, the $\angle$CRP will be $=\angle$CQU; therefore the triangles CUQ, CRP will be similar.

And in the same manner you may demonstrate, that the triangles SLQ, STP are similar.

Whence CR : PR : : QU : CU

and ST : PT : : QL : SL;

Suppose CS$=a$, CA$=b$, the sine of the angle FCO to its cosine as d to a; sine of the angle CAE to its cosine as c to a; and sine of the angle KSH to its cosine as e to a; put also PM$=$ y, CM$=x$ and QN$=z$;

then, MRP (FCO): MP : : cos. MRP : RM$=ay \div d$,

and PR$=(\text{PM}^2+\text{RM}^2)^{\frac{1}{2}}=y\cdot(a^2+d^2)^{\frac{1}{2}}\div d$;

also CR$=$CM$-$MR$=x-ay\div d$;

likewise QU$=z\cdot(a^2+d^2)^{\frac{1}{2}}\div d$,

and CU$=$CA$-$AN$-$NU$=b-az\div c-az\div d$;

but, it was shewn that, CR : PR : : QU : CU, that

 is

is $\frac{dx-ay}{d} : \frac{y}{d}\cdot(a^2+d^2)^{\frac{1}{2}} :: \frac{z}{d}(a^2+d^2)^{\frac{1}{2}} : \frac{bdc-az\cdot(d+c)}{dc}$.

Hence by multiplying means and extremes, &c. we have QN$=z=bc\cdot(dx-ay)\div\left(y\cdot(dc-a^2)+ax\cdot(d+c)\right)$.

In like manner you will find

ST $=a-x-ay\div e$, PT$=y\cdot(a^2+e^2)^{\frac{1}{2}}\div e$, QL$=z\cdot(a^2+e^2)^{\frac{1}{2}}\div e$; and SL$=$ AN—AS—NL$=a-b+az\cdot(e-c)\div ec$; but above it is shewn that ST : PT : : QL : SL, this proportion put into species gives QN $=z=$ $c\cdot(a-b)\cdot(ae-ex-ay)\div\left(y\cdot(ec+a^2)+ax\cdot(e-c)+a^2\cdot(c-e)\right)$; now by equating these two values of z, the resulting equation will be

$$\left(\overline{a-b}\cdot ce+\overline{ae-bc}\cdot d\right)\cdot x^2$$
$$+(a^2\cdot\overline{d+c-e}+dce)\cdot xy$$
$$+(a\cdot\overline{-a^2+cd}-bc\cdot\overline{e+d})\cdot y^2$$
$$+(abc\cdot\overline{d+e}-a^2e\cdot\overline{d+e})\cdot x$$
$$-(bc\cdot\overline{a^2-ed}+ae\cdot\overline{dc-a^2})\cdot y=0;$$

Where since x and y are only of two dimensions, it appears that the curve must be of the *first order* or a conic section.

Q. E. D.

The same answered by Mr. Colson.

This Gentleman, in his translation of Sir *Isaac Newton's* Fluxions and infinite Series, gives this problem with its solution, by way of a dialogue between a Master and his Scholar, to this effect.

Master. In the right line CS, I give you two points S and C.

Scholar. Then their distance SC$=m$, is also given.

Master. As likewise the two points P and Q out of the line SC.

Scholar. Then consequently the figure SPCQ is given in magnitude and specie; and producing PS and PC towards d and δ, I can take Sd = SQ and Cδ = CQ.

Master. Also I give you the indefinite right line AE in position passing through the point Q.

Scholar. Then the angles SQA and CQE are given, to which (producing SC both ways, if need be, to C and S), I can make the angles Sde and Cδf equal respectively, and that will determine the points e and f, or the lines Se = *a* and C f = *c*; and because de and δf are thereby known, I can continue de to g, so that dg = δf, and make the given line eg = *b*. Likewise, I can draw Ph and Pk parallel to ed and δf respectively, meeting SC in h and k, and because the triangle Phk will be given in magnitude and specie, I will make Pk = *d*, Ph = *e* and hk = *f*.

Master. Now let the given angles KSH and FCO be conceived to revolve about the given points or poles S and C.

Scholar. Then the lines SQ and PSd will move into another situation Sq and pSl, so that the angles QSq, dSl, and PSp will be equal.

Also the lines CQ and PCδ will obtain a new situation Cq and pCτ, so that the angles QCq, δCτ and PCp will be equal.

Master. And let Q the intersection of the lines SQ and CQ always move in the right line AE.

Scholar. Then the new point of intersection q is in AE; then the triangles QSq and dSl, as also QCq and δCτ are equal and similar; then dl = Qq = δλ and therefore gl = λf.

Master. What will be the nature of the curve described by the other point of intersection P?

Scholar. From the new point of intersection *p* to SC I will draw the lines *ph* and *pk*, parallel to Ph and Pk respectively.

Then

Then will the triangle phk be given in ſpecie, though not in magnitude, for it will be ſimilar to Phk.

Alſo the triangle Cpk will be the ſimilar to Crf.

And the indefinite line Ck $= x$, may be aſſumed for an abſciſs, and $pk = y$ may be the correſponding ordinate to the curve Pp.

Then, becauſe it is Ck : pk : : Cf : fr $= cy \div x =$ gl.

I can find le $=$ ge $-$ gl $= b - cy \div x$.

And becauſe of the ſimilar triangles phk, Phk,

it will be Pk : Ph : : pk : $ph = ey \div d$,

and Pk : hk : : pk : $hk = fy \div d$;

therefore Sh $=$ SC $-$ Ck $-$ $hk = m - x - fy \div d$,

but it is Sh : ph : : Se : le,

in ſpecie $(m - x - fy \div d) : ey \div d : : a : (b - cy \div x)$,

therefore $(m - x - fy \div d) \cdot (b - cy \div x) = aey \div d$,

or, $fcy^2 + (dc - ae - bf) \cdot xy - dcmy - bdx^2 + bdmx = 0$.

In which equation, becauſe the indeterminate quantities x and y ariſe only to two dimenſions, it ſhews that the curve deſcribed by the point P is a conic ſection.

Maſter. You have therefore ſolved the problem in general, but you ſhould now apply your ſolution to the ſeveral ſpecies of conic ſections in particular.

Scholar. That may eaſily be done in the following manner, make $(ae + bf - cd) \div c = 2p$, and the foregoing equation will become

$$fcy^2 - 2pcxy - dcmy - bdx^2 \div bdmx = 0.$$

and by extracting the ſquare root it will be $y =$

$$\frac{p}{f}x + \frac{dm}{2f} \pm \sqrt{\left(\frac{p^2}{f^2} + \frac{bd}{cf}\right) \cdot x^2 + \left(\frac{pdm}{f^2} - \frac{bdm}{fc}\right) \cdot x + \frac{d^2m^2}{4f^2}}.$$

Now here it is plain that if the term $\left(\frac{p^2}{f^2} + \frac{bd}{cf}\right) \cdot x^2$

were abſent or, or if $\frac{p^2}{f^2} + \frac{bd}{fc} = 0$, or

$p^2 +$

$p^2 \div f^2 = -bd \div fc$; that is, if the quantity $bd \div fc$, (changing its ſign) ſhould be equal to $p^2 \div f^2$, then the curve would be a *parabola*.

But if the ſame term were preſent and equal to ſome affirmative quantity, that is, if $p^2 \div f^2 + bd \div fc$ be affirmative (which will always be the caſe when $bd \div fc$ is affirmative, or if it be negative and leſs than $p^2 \div f^2$) the curve will be an *hyperbola*.

Laſtly, if the ſame term were preſent and negative, (which can only be when $bd \div fc$ is negative and greater than $p^2 \div f^2$) the curve will be an *ellipſis* or a *circle*.

The ſame anſwered from Emerſon's *Algebra.*

Draw QN and PM perpendicular to SC; and put AS $= a$, SC $= b$, tang. ∠ SAQ $= t$, tang. ∠ PSQ $= p$, tang. ∠ PCQ $= q$, SN $= v$, SM $= x$, PM $= y$, and CN $= b - v$, CM $= b - x$.

Then by trigonometry $1 : a + v :: t : ta + tv =$ QN,

and $v : 1 :: ta + tv : (ta + tv) \div v =$ tang. QSN;

alſo $x : 1 :: y : y \div x =$ tang. PSM,

and $b - v : 1 :: ta + tv : (ta + tv) \div b - v =$ tang. QCN,

and $b - x : 1 :: y : y \div (b - x) =$ tang. PCM.

but by trig. I. 8, $1 - \frac{ta + tv}{vx} y : 1 :: \frac{ta + tv}{v} + \frac{y}{x} : p$,

and $1 - \frac{ta + tv}{b - v} \cdot \frac{y}{b - x} : 1 :: \frac{ta + tv}{b - v} + \frac{y}{b - x} : q$;

Hence by multiplying means and extremes, and afterwards reducing, we have theſe two equations,

$$pvx - ptay - ptvy = tax + tvx + vy,$$

and $(tv + qv) \cdot (b - x) + tqvy - vy = qb \cdot (b - x) - taqy + tax - tab - by$.

From theſe two laſt equations we get

$$v = \frac{qb^2 - qbx + taqy - tab - tax - by}{qb - qx + tqy + tb - tx - y} = \frac{tax + ptay}{px - tx - pty - y}.$$

And

And by fubftituting for the known compound quantities we have $\frac{cx+dy+f}{-gx+hy+l}=\frac{tax+tapy}{nx-sy}$, this reduced the refulting equation is

$$\left.\begin{matrix}taph\\+sd\end{matrix}\right\}y^2\left.\begin{matrix}-tag\\-cn\end{matrix}\right\}x^2\left.\begin{matrix}-tapg\\+tah\\+sc\\-dn\end{matrix}\right\}xy\left.\begin{matrix}+tal\\-fn\end{matrix}\right\}x\left.\begin{matrix}+tapl\\+sf\end{matrix}\right\}y=0.$$

which being an equation of two dimenfions, the curve will be a *conic fection.*

ARTICLE XVII.

A COLLECTION OF PROBLEMS.

To be anfwered in Number IV.

I. QUESTION 29. *by Mr.* Ra. Simpfon.

SUPPOSE one penny had been lent at compound intereft at 5 *per cent.* in the firft year of the Chriftian *era,* or birth of Chrift, and fo continued to this prefent year 1796. *Query,* the exact amount thereof?

II. QUESTION 30. *by the Rev. Mr.* L. Evans.

Three perfons join ftock A, B and C; A put in £.150, for 14 months, B put in a certain fum for 12 months, C put in a certain fum for a certain time, fo that their ftock and gain were £.475, of which A took £.195, B, £.153, and C, £.127. I demand B's ftock, alfo C's ftock and time?

III. QUESTION 31. *by Mr. T. Bulmer, Teacher of the Mathematics, at Sunderland.*

A wants to purchafe of B an annuity of £.25 *per annum*

annum, payable half-yearly, to continue 27 years; the purchafe money to be paid by A to B half-yearly, reckoning each at compound intereft at $4\frac{1}{2}$ *per cent*. *Query*, how much muft A pay B every half-year to make up the purchafe money in 11 years?

IV. QUESTION 32. *by Mr.* Jon. Mabbott.

At page 418, *Emerfon's Algebra*, is given the equation $y^4+2dy^3+(d^2-4r^2)\cdot y^2-4r^2dy+r^2b^2=0$; it is required to find the root y?

V. QUESTION 33. *by Mr.* Burdon.

Given the bafe of a plane triangle, the angle made by one fide and the bafe, and the difference between the other fide and the perpendicular, to conftruct it?

VI. QUESTION 34. *by Mr.* Louis Hill, *Rowley*.

Given the ratio of the bung and head diameter as one to three-fourths, and the difference between the femi-length and diagonal 8 inches, of a cafk of the 2d variety; to find the dimenfions when it holds 98 ale gallons?

VII. QUESTION 35. *by Mr.* Hill.

Given the fide of a fpherical fquare $=28^\circ\ 16'$; to find its area, and the radius of its infcribed circle?

VIII. QUESTION 36. *by* Nauticus.

Given the latitude and longitude of three places on the earth; to find a fourth place, fuch, that the fum of the diftances from it to the other three may be the leaft poffible, fuppofing the earth a perfect fphere?

IX.

IX. QUESTION 37. *by* Appolonius, junior.

To determine a point in the base of a plane triangle, such, that if perpendiculars be drawn from it to the other two sides, the area of the trapezium so formed may be a *maximum*?

X. QUESTION 38. *by the same.*

To determine a point in a curve of any order given by position, such, that if two lines be drawn from it to make given angles with two parallel lines given also by position, the sum or difference of their squares may be equal to a given square?

XI. QUESTION 39. *by Mr.* O. G. Gregory.

The area of the whole figure of the versed sines of a certain circle is 2827·4337, from hence it is proposed to determine the difference between the areas of the least equilateral triangle, and the least rectangled triangle that circumscribe the circle?

XII. QUESTION 40. *by Mr.* John Lowry.

Given the area of a plane triangle, the sum of the sides, and the rectangle contained by the segments of the base made by the point of contact of the inscribed circle, to construct it?

XIII. QUESTION 41. *by Mr.* Lowry.

Given the vertical angle and the difference of the sides of a plane triangle to construct it, when the rectangle of the sides has to the rectangle of the segments of the base made by the perpendicular a given ratio?

XIV. QUESTION 42. *by Mr.* Mabbot.

Required the sum of the infinite series

$$\frac{1}{9\cdot11\cdot12\cdot28}-\frac{1}{10\cdot12\cdot14\cdot30}+\frac{1}{11\cdot13\cdot16\cdot32}-\&c.$$

by Mr. *Sterling's* method?

XV.

XV. QUESTION 43. *by* Mr. W. Pearſon.

Suppoſe a comet containing the ſame quantity of matter as the moon, to paſs between the moon and earth equally diſtant from the two bodies, to find how high it will raiſe the waters in the ocean?

XVI. QUESTION 44. *by* Mydorgius.

Let APBQCRDSET, &c. (Fig. 74, Plate 4.) be an ellipſis, and *abcdef*, &c. *pqrstv*, &c. two polygons of an equal number of ſides deſcribed about the ſame, ſuch, that the ſides of each polygon may be biſected by their reſpective points of contact, A, P, B, Q, C, R, D, S, E, T, &c. *i. e.* ſuch that, $aA = Ab$, $bB = Bc$, $cC = Cd$, &c. and $pP = Pq$, $qQ = Qr$, $rR = Rs$, &c. Then I ſay, that the ſum of the ſquares of the ſides of one polygon will be equal to the ſum of the ſquares of the ſides of the other polygon; that is, $ab^2 + bc^2 + cd^2 + de^2 +$ &c. $=$ $pq^2 + qr^2 + rs^2 + st^2 +$ &c. And if the lines AB, BC, CD, DE, &c. PQ, QR, RS, ST, &c. be drawn, then I ſay, that $AB^2 + BC^2 + CD^2 +$ &c. $= PQ^2 + QR^2 + RS^2 +$ &c.

Alſo, if O be the centre of the ellipſis and the lines OA, OP, OB, OQ, OC, OR, &c. be drawn I ſay that $OA^2 + OB^2 + OC^2 + OD^2 +$ &c. $=$ $OP^2 + OQ^2 + OR^2 + OS^2 +$ &c.

And, if the lines Oa, Op, Ob, Oq, Oc, Or, &c. be drawn, then I ſay that $Oa^2 + Ob^2 + Oc^2 + Od^2$ $+$ &c. $= Op^2 + Oq^2 + Or^2 + Os^2 +$ &c. Required the demonſtration?

XVIII. QUESTION 45. *by* Pappus, junior.

Let there be an ellipſis whoſe tranſverſe axis is AB, and foci D, E; produce DA to F; and let DF be equal to AB; upon DF let there be a ſeme-

the circumference E two lines CE, HE meeting the ſame again in D and G; I ſay that EC : CD : : EH : HG.

PROP. IX.

If in AB the diameter of a circle be taken any point C, and CD be drawn meeting the circumference in D and E, and from the point D be drawn DF perpendicular to CD, which meets the diameter AB in F and the circumference in G; then I ſay that DC : CE : : DF : FG.

PROP. X.

If in AB the diameter of a circle two points C and D be taken ſuch that AC : CB : : AD : DB, and through the centre E a perpendicular to AB be drawn, and from C a line be drawn to meet the ſame in F, and if through D any line DG be drawn to meet the circle in G and H, and from the point G be drawn GK the ſame ſide of DG, as F is of the diameter AB to make the angle DGK equal to the angle CFE, and let the line GK meet the circle in L and the line CF in M; then I ſay that GM : ML : : DG : DH.

ARTICLE XIX.

Containing four Propoſitions from *Stewart's General Theorems.*

To be demonſtrated in Number III.

PROP. XI. THEO. VIII.

LET there be any number of given points, two points may be found, ſuch, that if from all the given points and the two points found there be drawn right lines to any point, twice the ſum of the ſquares of the lines drawn from the given points, will be equal to the multiple by the number of the given points of the ſum of the ſquares of the lines drawn from the two points found.

PROP.

PROP. XII. THEO. IX.

Let there be any number of given points, and let a, b, c, &c. be given magnitudes, as many in number as there are given points; two points may be found, ſuch, that if from all the given points and the two points found there be drawn right lines to any point, the ſquare of the line drawn from one of the given points, together with the ſpace to which the ſquare of the line drawn from another of the given points has the ſame ratio that a has to b, together with the ſpace to which the ſquare of the line drawn from an other of the given points has the ſame ratio that a has to c, and ſo on, will be equal to the ſpace to which the ſum of the ſquares of the lines drawn from the two points found has the ſame ratio that twice a has to the ſum of a,b,c,&c.

PROP. XIII. THEO. X.

Let there be any number of right lines given by poſition, and parallel to each other; a right line may be found parallel to the lines given by poſition, ſuch, that if from any point there be drawn a perpendicular to the right lines given by poſition and to the line found, the ſum of the ſquares of the lines intercepted between the point and the right lines given by poſition, will be equal to the multiple of the ſquare of the line intercepted between the point and the right line found, by the number of the right lines given by poſition, together with a given ſpace.

PROP. XIV. THEO. XI.

Let there be any number of right lines interſecting each other in one point, and making all the angles round the point of interſection equal; and from any point let there be drawn perpendiculars to the right lines, and likewiſe let there be drawn a right line to the point of interſection: twice the ſum of the ſquares of the perpendiculars will be equal to the multiple of the ſquare of the line drawn to the point of interſection by the number of the lines.

ARTICLE

ARTICLE XX.

Containing ſix Propoſitions from *Lawſon* on the *Ancient Analyſis.*

To be demonſtrated in Number IV.

PROP. XI.

IF from any point C, in the diameter of a circle produced a perpendicular be raiſed and from any point D in the ſame a line be drawn to cut the circle in E and F; then I ſay the rectangle EDF is equal to the rectangle A C B together with the ſquare of CD.

PROP. XII.

If from any point C in the diameter of a circle produced, a perpendicular be raiſed and thereon, CD be taken, whoſe ſquare is equal to the rectangle ACB, and CE be put equal CD, and from any point in DE as H a line be drawn to cut the circle in F and G; then I ſay twice the rectangle FHG is equal to the ſum of the ſquares of HD and HE.

PROP. XIII.

If in AB, the diameter of a circle, two points C and D be ſo taken that C being without, and D either within or without the circle, the ſquare of CD be equal to the rectangle ACB, and from C a perpendicular to AB erected, and any line drawn through D to cut the ſame in G and the circle in E and F; then I ſay the ſquare of GD will be equal to the rectangle EGF.

The converſe of this is alſo true, which is this:

If GC be perpendicular to AB the diameter of a circle, and meets it without the circle in C, and if from G a line be drawn to cut the circle in E and

F,

F, and the diameter either within or without in D, and the ſquare of GD be equal to the rectangle EG F; then I ſay the ſquare of CD will be equal to the rectangle ACB.

PROP. XIV.

Things remaining as in the laſt propoſition, if the perpendiculars Eg and FH be demitted; then I ſay that the rectangle gCH is equal to the ſquare of CD.

PROP. XV.

If from C, any point in the diameter of a circle AB produced a tangent be drawn, and from the point of contact D a perpendicular to the diameter DE be demitted; then I ſay that AC : CB :: AE : EB.

Or converſely thus:

If in AB, the diameter of a circle be taken two points, C and E, ſuch that AC : CB :: AE : EB, and from E a perpendicular ED raiſed, and CD drawn; then I ſay CD touches the circle in D.

Or thus:

If in AB the diameter of a circle produced a point C be taken, and therefrom a tangent as CD be drawn, and in the diameter a point E be taken, ſuch that AC : CB :: AE : EB; then I ſay ED being drawn will be perpendicular to the diameter AB.

PROP. XVI.

Let AB be any chord in a circle, and CD another cutting the former in E, CB being joined, from D draw DF parallel to CB to meet AB in F; I ſay the rectangle AEF is equal to the ſquare of DE.

PROP. XVII.

If ABC be a triangle inſcribed in a circle whoſe ſides CA anb CB are equal, and the rectangle CBD equal to the ſquare of AB, and let AE be any line cutting

cutting CB in F, and the circle again in E, and from E let a parallel to AB be drawn to meet CB in G; then I ſay that the rectangle CFG : BF² :: CG : BD.

ARTICLE XXI.

Containing four Propoſitions from *Stewart's General Theorems*.

To be demonſtrated in Number IV.

PROP. XV. THEO. XII.

Let there be any number of right lines given by poſition, interſecting each other in a point; two right lines may be found that will be given by poſition, ſuch, that if from any point there be drawn perpendiculars to all the right lines given by poſition, and likewiſe perpendiculars, to the two right lines found, twice the ſum of the ſquares of the perpendiculars drawn to the right lines given by poſition, will be equal to the multiple of the ſum of the ſquares of the perpendiculars drawn to the two lines found by the number of the lines given by poſition.

PROP. XVI. THEO. XIII.

Let there be any number of right lines given by poſition, that are neither all parallel, nor interſecting each other in one point; two right lines may be found that will be given by poſition, ſuch, that if from any point there be drawn perpendiculars to the right lines given by poſition, and likewiſe perpendiculars to the two right lines found, twice the ſum of the ſquares of the perpendiculars drawn to the right lines given by poſition, will be equal to the multiple of the ſum of the ſquares of the perpendiculars

diculars drawn to the two right lines found by the number of the right lines given by pofition together with a given fpace.

PROP. XVII. THEO. XIV.

Let there be any number greater than three of right lines given by pofition; three right lines may be found that will be given by pofition fuch, that if from any point there be drawn perpendiculars to all the right lines given by pofition, and likewife to the three lines found, thrice the fum of the fquares of the perpendiculars drawn to the right lines given by pofition, will be equal to the multiple of the fum of the fquares of the perpendiculars drawn to the three lines found by the number of the lines given by pofition.

PROP. XVIII. THEO. XV.

Let there be any number of right lines given by pofition, and parallel to each other; and let $a, b, c,$ &c. be given magnitudes as many in number as there are right lines given by pofition; a right line may be found parallel to the right lines given by pofition fuch, that if from any point there be drawn a perpendicular to the right lines given by pofition, and likewife to the right line found, the fquare of the fegment intercepted between the point and one of the right lines given by pofition, together with the fpace to which the fquare of the fegment intercepted between the point and another of the lines given by pofition has the fame ratio that a has to b, together with the fpace to which the fquare of the fegment intercepted between the point and another of the lines given by pofition has the fame ratio that a has to c, and fo on, will be equal to the fpace to which the fquare of the fegment intercepted between the point and the line found, has the fame ratio that a has *to the fum of* $a, b, c,$ &c. together with a given *fpace.*

ERRATAS.

Through the EDITOR's being at a distance from the PRESS, a number of *Erratas* have occured in printing the REPOSITORY, which the reader is desired to correct the following.

Page	Line	For	Read
78	4	$(\dot{x} - \dot{y})$	$(x - y)$
90	3	Trig.) = s	Trig.) s
	8	Fig. 63	Fig. 62
92	31	ICP	ICD
93	34	ABQS	ABGS
94	6	s. BC	s. BG
95	29	*THEOREM*	*PROP.*
96	2	B	*b*
105	6	&c. and	&c. x being $= -1$ and
	15	x being $= 1$	x being $= -1$
107	16	x being $= 1$	x being $= -1$
108	14	$z+4 \cdot z+5$	$z+5 \cdot z+6$
110	7	$y^{\frac{1}{4}}$	$y^{\frac{1}{2}}$
114	8	$2^{\frac{3}{2}}$	$2^{\frac{5}{2}}$
	13	$a^{\frac{5}{2}}$	$a^{\frac{3}{2}}$
126	13	rect. FCK	rect. ECK
	24	ACB — rect.	ACB = rect.
128	22	$(\dot{x} + \gamma y)$	$(x + \gamma y)$
	23	in the num. γy	γy
		in the denom. $\dot{x}$	x
130	30	draw DX	draw to DX

REMARKS

TO

CORRESPONDENTS.

THE EDITOR underſtands Mr. GREGORY's *New Treatiſe on the Sliding Rule*, and Mr. LOWRY's *New Tables on Gauging*, are both to be publiſhed by Subſcription as ſoon as a ſufficient Number are ſubſcribed for, the EDITOR himſelf takes four Copies of each; thoſe, therefore, who wiſh to become Encouragers of theſe Works, will pleaſe to ſignify the ſame to the EDITOR; who will communicate the Subſcriptions to the reſpective Authors.

The EDITOR returns his ſincere Thanks to Mr. BULMER, for his engaging to take twelve Copies of the Repoſitory—he hopes others will follow the like Example.

He alſo begs leave to inform them that No. III. will be ready on the 1ſt of March, 1797, and all letters for its uſe muſt come to hand by the 1ſt of November except letters containing Demonſtrations to *Lawſon's*, or *Stewart's* Propoſitions and Anſwers to the Philoſophical Queſtions propoſed in this Number, which will be in time by the middle of November.

THE

MATHEMATICAL REPOSITORY.

ARTICLE XXII.

On the Resolution of Indeterminate Problems. *By* John Leslie, *A. M.*

rom the Second Volume of Edinburgh Philosophical Transactions.

T is a fundamental principle in Algebra, that a problem admits of solution, when the number of ependent equations is equal to that of the un- own quantities. If simple expressions only oc- , the answers will always be found in numbers, ier whole or fractional. But if the higher func- is be concerned, the values of the unknown ntities will commonly be involved in surds, ich it is impossible to exhibit on any arithmeti- scale, and to which we can only make a repeated roximation. Hence the origin of that branch of lysis which is employed in the investigation of se problems, where the number of unknown antities exceed that of the proposed equations, where the values are required in whole or frac- nal numbers. The subject is not merely an object curiosity; it can be applied with advantage to the her calculus. Yet the doctrine of indeterminate iations has been seldom treated in a form equally tematic with the other parts of algebra. The so- ions commonly given are devoid of uniformity, l often require a variety of assumptions. The

Q object

object of this paper is to resolve the complicated expressions which we obtain in the solution of indeterminate problems, into simple equations, and to do so, without framing a number of assumptions, by help of a single principle, which, though extremely simple, admits of a very extensive application.

Let $A \times B$ be any compound quantity equal to another $C \times D$, and let m be any rational number assumed at pleasure; it is manifest that, taking equimultiples, $A \times mB = C \times mD$. If therefore, we suppose, that $A = mD$, it must follow, that $mB = C$, or $B = C \div m$. Thus two equations of a lower dimension are obtained. If these be capable of a farther decomposition, we may assume the multiples n and p, and form four equations still more simple. By the repeated application of this principle, an higher equation, if it admits of divisors, will be resolved into those of the first order, the number of which will be one greater than that of the multiples assumed. Hence the number of simple equations into which a compound expression can be resolved, is equal to the sum of the exponents of the unknown quantities in the highest term. Wherefore a problem can be solved by the application of this principle only, when the *aggregate sum*, formed by the addition of the exponents in the highest terms of the several equations proposed, is *at least* equal to the number of the unknown quantities, together with that of the assumed multiples.

We shall illustrate the mode of applying our principle, in the solution of some of the more general and useful problems connected with this branch of analysis.

PROBLEM I.

Let it be required to find two rational numbers, the difference of the squares of which shall be a given number.

Let

Let the given number be the product of a and b; then by hypothesis, $x^2 - y^2 = ab$; but these compound quantities admit of an easy resolution, for $(x+y)\cdot(x-y) = a \times b$. If therefore we suppose $x+y = ma$, we shall obtain $x-y = b \div m$, where m is arbitrary; and if rational, x and y must also be rational. Transposing the first equation, $x = ma - y$, and reducing the second, $mx - my = b$, and transposing $mx = b + my$, and therefore, $x = (b+my) \div m$; whence by equality $(b+my) \div m = ma - y$, and reducing, $b + my = m^2 a - my$, and transposing $2my = m^2 a - b$, whence $y = (m^2 a - b) \div 2m$; but $x = ma - y$, consequently $x = (m^2 b + a) \div 2m$. If $m = 1$; then $x = \frac{1}{2}(b+a)$, and $y = \frac{1}{2}(b-a)$.

Suppose it were required to find a number which, increased or diminished by 10, would produce squares. It is obvious that the number may be denoted, either by $x^2 - 10$, or $y^2 + 10$; whence $x^2 - 10 = y^2 + 10$, and transposing $x^2 - y^2 = 5 \times 4$, and applying the above formulæ, $x = (5m^2 + 4) \div 2m$; if $m = 2$, then $x = 6$ and the required number 26.

PROBLEM II.

To find two numbers, the sum of the squares of which shall be equal to the sum of two given squares.

By hypothesis $x^2 + y^2 = a^2 + b^2$, and transposing $x^2 - a^2 = b^2 - y^2$, and by resolving into factors, $(x+a)\cdot(x-a) = (b+y)\cdot(b-y)$; whence, by substitution, $x + a = mb - my$, and $x - a = (b+y) \div m$. Transposing the first equation, $x = mb - my - a$; reducing the second, $mx - ma = b + y$, and transposing, $mx = ma + b + y$, and therefore $x = (ma + b + y) \div m$; whence $(ma + b + y) \div m = mb - my - a$, and $ma + b + y = m^2 b - m^2 y - ma$, and transposing $m^2 y + y = m^2 b - 2ma - b$,

$-b$, that is, $y = (m^2b - 2ma - b) \div (m^2 +$
But $x = mb - my - a$, and substituting, x
$(m^2a + 2mb - a) \div (m^2 + 1)$. Thus, if $a =$
and $b = 10$, and $m = 2$; then $y = (4 \cdot 10 -$
$- 10) \div 5 = 2$, and $x = (4 \cdot 5 + 4 \cdot 10 - 5)$
$= 11$; but $(11)^2 + (2)^2 = 125 = (10)^2 +$

Cor. If $b = o$, we shall obtain two squares, sum of which shall be a given square; for $y =$
$2ma \div (m^2 + 1)$, or $+ 2ma \div (m^2 + 1)$, and
$(m^2a - a) \div (m^2 + 1)$. Thus, if $a = 10$,
$m = 2$, then $y = 4 \cdot 10 \div 5 = 8$, and $x = ($
$- 10) \div 5 = 6$, but $64 + 36 = 100$.

PROBLEM III.

To find two rational numbers, the square which, together with any given multiple of product, shall be equal to a given square.

By hypothesis, $x^2 + y^2 + bxy = a^2$, and transposing $x^2 + bxy = a^2 - y^2$, and resolving factors, $x \cdot (x + by) = (a + y) \cdot (a - y)$; whence by assumption, $x + by = ma - my$, and $x = ($
$y) \div m$. Transposing the first equation, $x =$
$- my - by$, consequently, $(a + y) \div m = m$
$my - by$, or $a + y = m^2a - m^2y - mby$, and again transposing, $m^2y + mby + y = m^2a - a$; whence
$a \cdot (m^2 - 1) \div (m^2 + mb + 1)$. But $x = (a +$
m, wherefore $x = a \cdot (2m + b) \div (m^2 + mb$

Suppose $a = 22$, $b = 3$, and $m = 2$, then
$22 \cdot (4 + 3) \div (4 + 6 + 1) = 14$, and $y =$
$(4 - 1) \div (4 + 6 + 1 (= 6$. But $196 +$
$252 = 484 = (22)^2$.

Cor. If $b = 1$, the hypothesis will be $x^2 +$
$xy = a$; and $x = a \cdot (2m + 1) \div (m^2 + m$
and $y = a \cdot (m^2 - 1) \div (m^2 + m + 1)$. Thus
$a = 13$, and $m = 3$, then $x = 13 \cdot (6 + 1) \div$
$3 + 1) = 7$, and $y = 13 \cdot (9 - 1) \div (9 + 3$
$= 8$. But $49 + 64 + 56 = 169 = (13)^2$.

Pl

PROBLEM IV.

To find two numbers, ſuch, that each, increaſed by unit, ſhall be a ſquare, and their ſum increaſed by unit, a given ſquare.

Let the numbers be denoted by $x^2 - 1$ and $y^2 - 1$, and the firſt condition will be obſerved. The laſt requires, that $x^2 - 1 + y^2 - 1 + 1$, or $x^2 + y^2 - 1 = a^2$. By tranſpoſition, $x^2 - 1 = a^2 - y^2$, and by reſolution, $(x + 1) \cdot (x - 1) = (a + y) \cdot (a - y)$; whence $x + 1 = ma - my$, and $mx - m = a + y$. Tranſpoſing the firſt equation $x = ma - my - 1$; and tranſpoſing the ſecond, $mx = a + y + m$, and dividing, $x = (a + y + m) \div m$, whence $(a + y + m) \div m = ma - my - 1$, and reducing $a + y + m = m^2a - m^2y - m$, or $m^2y + y = m^2a - 2m - a$, and therefore $y = (m^2a - 2m - a) \div (m^2 + 1)$. But $x = (a + y + m) \div m$, whence $x = (m^2 + 2ma - 1) \div (m^2 + 1)$.

Suppoſe $a = 8$, and $m = 2$, then $x = (4 + 32 - 1) \div (4 + 1) = 7$, and $y = (4 \cdot 8 - 4 - 8) \div (4 + 1) = 4$, and the numbers are 48 and 15; but $48 + 15 + 1 = 64 = (8)^2$.

ARTICLE XXIII.

Atwood's *Inveſtigations on Watch Balances.*

(Continued from page 89.)

THE poſition of the centre of gyration may be always determined when the figure of the vibrating body is regular, by calculating the ſum of the products which ariſe from multiplying each particle into the ſquare of its diſtance from the axis of motion, and dividing the ſum by the weight of the vibrating body; the ſquare root of the reſult will be

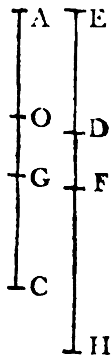
the diftance of the centre of gyration from the axis of motion. When the figure of the vibrating body is irregular, recourfe may be had to experimental * methods, in order to determine the pofition of the centre of gyration.

Let the radius of the balance CA or CO $= r$, (fig. 79.) the femi-arc BO $= b$; let the fpring's elaftic force, acting on the circumference of the balance, when wound to any given angle OCD from the quiefcent pofition be $=$ P, and let the arc OD $= a$; the weight of the balance, and the parts which vibrate with it $=$ W; the diftance of the centre of gyration from the axis of motion CG $= g$. Thefe notations being premifed, the refiftance of inertia by which the mafs contained in the balance oppofes the communication of motion to the circumference is $Wg^2 \div r^2$: and confequently the force which accelerates the circumference at the angular diftance OCD from the quiefcent pofition is $Pr^2 \div Wg^2$. This quantity remaining invariably the fame, while the balance defcribes the arc of vibration BOE, may be denoted by the letter F, fo that $F = Pr^2 \div Wg^2$; fuppofe the radius CA commencing a vibration from the point B to have defcribed the arc BH, and let OH $= x$; fince the force which accelerates the circumference at the angular diftance from quiefcence OD is $=$ F, and the forces of acceleration are fuppofed to vary in the proportion of the angular diftances from the quiefcent point O, the force which accelerates the circumference of the balance at the point H will be $= Fx \div a$; let u be the fpace through which a body falls freely from reft by the acceleration of gravity to acquire the velocity of the circumference, at the point H; the principles of

* Treatife on the Rectilinear Motion and Rotation of Bodies, page 226 and 301.

accele-

acceleration give this equation*, $\dot{u} = - Fx\dot{x} \div a$; and taking the fluents while x decreaſes from b to x, $u = F \cdot (b^2 - x^2) \div 2a$: if therefore l is made $=$ 193 inches, being the ſpace which bodies falling freely from reſt by the force of gravity near the earth's ſurface deſcribe in one ſecond of time, the velocity of the circumference, when the extremity



* Newtonii Principia, vol. I. prop. XXXIX. Let a body deſcribe the line AC by the acceleration of a force varying in any ratio of the diſtances from a centre C. Let another body deſcribe the line EH by the acceleration of a conſtant or uniform force. Suppoſe the velocity at O to be equal to the velocity at D, and let OG and DF be the evaneſcent ſpaces, or increments of ſpace in which equal velocities are generated; ſo that ED may repreſent a line through which a body muſt fall from reſt by the acceleration of the conſtant or uniform force, to acquire the velocity of the other body at O. It is to be proved that the increment of ſpace OG is to the increment of ſpace DF, as the force of acceleration at D to the force of acceleration at O. Let the former of theſe forces, *i. e.* at D be denoted by G, and the latter force at O by H. Let ED $= u$ and let AO $= x$. Alſo let DF $= \dot{u}$, and OG $= \dot{x}$. Becauſe the increments of velocity are always as the forces of acceleration and the elementary times in which they act jointly, it follows, that when the increments of velocity are equal, the forces are in the inverſe ratio of the elementary times in which they act, that is (the velocities of deſcribing the evaneſcent ſpaces OG, DF being equal by the ſuppoſition), the forces are in the inverſe ratio of thoſe ſpaces; and conſequently the force at D(G) is to the force at O(H) as OG to DF; that is, according to the preceeding notation, $G : H :: \dot{x} : \dot{u}$ or $\dot{u} = H\dot{x} \div G$. The conſtant force G being aſſumed equal to that of gravity, may be denoted by any conſtant quantity, ſuch as unity. By ſubſtituting therefore 1 for G, the equation will become $\dot{u} = H\dot{x}$. In this expreſſion the lines u and x are ſuppoſed to increaſe together, but if u increaſes while x decreaſes, the ſigns of the variations $\dot{u}$ and $\dot{x}$ will be contrary; in which caſe the equation will become $\dot{u} = - H\dot{x}$.

A of

A of the index CA has arrived at the point H, will be $= \sqrt{2lF \div a} \times \sqrt{b^2 - x^2}$. Let t represent the time in which the circumference describes the arc BH; then will $\dot{t} = \sqrt{a \div 2lF} \times - \dot{x} \div \sqrt{b^2 - x^2}$; and $t = \sqrt{a \div 2lF} \times$ into a circular arc, of which the cosine $= x \div b$ to radius $= 1$, which is the time of describing the arc BH expressed in parts of a second; when $x = 0$, that is when the circumference has described the entire semiarc BO, the circular arc of which the cosine $= x \div b$ is a quadrant of a circle to radius equal 1. Let $p = 3{\cdot}14159$, &c. The time t, of describing the semiarc $BO = \sqrt{a \div 2lF} \times \frac{1}{2}p = \sqrt{p^2 a \div 8lF}$. In this expression for the time of a semivibration, the letter a denotes the length of the arc OD (fig. 79.); if this arc should be expressed by a number of degrees c°, a will then $= prc^\circ \div 180^\circ$; and this quantity being substituted for a, the time of a semivibration will be $t = \sqrt{p^3 rc^\circ \div (8lF \times 180^\circ)}$; if instead of F, its value $Pr^2 \div Wg^2$ is substituted in the equation $t = \sqrt{p^3 rc^\circ \div (8lF \times 180^\circ)}$, the time of a semivibration will be $t = \sqrt{Wp^3 g^2 c^\circ \div (8Prl \times 180^\circ)}$. Let the given arc c° be $= 90^\circ$; in this case $t = \sqrt{Wp^3 g^2 \div 16Prl}$. These are expressions for the time of a semivibration, whatever may be the figure of the balance, the other conditions remaining the same as they have been above stated. If the balance should be a cylindrical plate it is known that the distance of the centre of gyration from the axis is to the radius as 1 to $\sqrt{2}$; wherefore in this case $g^2 = \frac{1}{2}r^2$; and the time of a semivibration, or $t = \sqrt{Wp^3 r \div 32Pl}$*. It is observable that the semi-

* The balances of watches are usually of such a form as to place the centre of gyration nearly at the same distance from the axis, as if the figures were cylindrical plates of uniform thickness and density. If it should be required to obtain from theory the time of a

arc

arc of vibration BO $= b$, does not enter into these expressions for the time of a semivibration; if therefore instead of the semi-arc BO, an arc of any other length LO, terminating in the point of quiescence O,

balance's vibration precisely exact, it would be necessary to calculate rigidly the position of the centre of gyration from the dimensions of each part of the balance, and whatever vibrates with it. But in cases merely illustrative of the general theorems for ascertaining the times of vibration, it is unnecessary to enter into prolix and troublesome calculations depending on the form of any particular balance; since by assuming it as a cylindrical plate, the time of a vibration will not differ materially from that which would be the result of the correct investigation.

Being desirous of comparing the time of vibration, as deduced from the theory of motion, with the actual vibration of a watch balance, I requested Mr. *Earnshaw* (the excellent performance of whose time-keepers is well known) to make the experiments from which the necessary data for this calculation are derived. These experiments were made on the balance of a watch constructed by Mr. *Kendal*, on Mr. *Harrison's* principles, and is the instrument which Captain *Cook* took out with him during his last voyage to the South Seas. The results are underneath.

Diameter of the balance - - - -	2¼ inches
Weight of the balance, and parts which vibrate with it	42 grains
Weight applied to the circumference of the balance, which counterpoises the force of the spiral spring when the balance is wound through an angle of 180°	48 grains
The weight which counterpoises the spring's force when the balance is wound to 90° from quiescence is	24 grains

These determinations give the following substitutions in the expression for the time of a semivibration $t = \sqrt{Wp^2r \div 32Pl}$.

Namely, $W = 42$ grains = the weight of the balance, including the parts which vibrate with it.

$P = 24$ grains = the force at the circumference of the balance which counterpoises the force of the spring when wound to the distance 90°.

$r = 1{\cdot}125$ inches and parts = the radius of the balance.

$l = 193$ inches = the space described in one second of time by bodies which descend freely from rest by the acceleration of gravity.

$p = 3{\cdot}14159$, &c. = the circumference of a circle of which the diameter $= 1$; the time of a semivibration $t =$

$\sqrt{(42 +}$

O, (fig. 79.) fhould be fubftituted in the preceding inveftigation, the time of defcribing LO would be ftill $= \sqrt{ap^2 \div 8lF}$ or $\sqrt{p^3rc^o \div (8lF \times 180^o)}$ equal to the time of defcribing the other femiarc BO; confequently, whether the balance vibrates in the largeft or fmalleft arcs, the times of vibration will be the fame.

From the preceding inveftigations it appears, that when the force by which the circumference of the balance is accelerated at the given angular diftance c^o from the quiefcent pofition is $=$ F, the time of a femivibration $t = \sqrt{p^3rc^o \div (8lF \times 180^o)}$; and converfely, when the time of a femivibration is $= t$, the force which accelerates the circumference at the given angular diftance c^o from the quiefcent pofition, that is, $F = p^3rc^o \div (8lt^2 \times 180^o)$.

(To be continued.)

ARTICLE XXIV.

LUCUBRATIONS IN SPHERICS.

By Mr. JOHN LOWRY.

(Continued from page 99.)

PROP. XXV. THEOREM. Fig. 87, Plate 6.

IF the bafe AB of any fpherical triangle ABC be produced both ways, fo that AD may be equal to the fide AC, and BE equal to the fide BC, and O be the pole of a circle defcribed through the three points

pts. of a fecond.

$\sqrt{([illegible] \times 3{\cdot}14159)^2 \times 1{\cdot}125) \div (32 \times 24 \times 193)} =$ 0·0994

The balance, when adjufted to mean time, makes [illegible] vibrations in a fecond; the actual time of a femivibration is therefore - - - - - 0·1000

Difference between the actual time and the time by the calculation - - - - - - 0·0006.

D

D, C, E; then I ſay the great circle deſcribed through the points OC, will biſect the vertical angle ACB.

Demon. Draw the great circles OD, OE, DC, CE; and from O, upon the baſe AB, demit the perpendicular arch OP.

Becauſe O is the pole of the circle DCE, the △'s DOC, EOC, will be iſoceles, and by hyp. the △'s DAC, EBC, are iſoceles; therefore ∠ODC = OCD and ∠ADC = ∠ACD;

therefore ∠ ODA = ∠ OCA;

in like manner ∠ OEB = ∠ OCB:

but the right angled triangles OPD, OEP, having OD = OE and OP common to both, will alſo have ∠ ODP (ODA) = ∠ OEP (OEB);

therefore ∠ OCA = ∠ OCB;

therefore the arch OQC biſects the vertical angle.

Q. E. D.

Cor. 1. If the baſe AB be biſected in I; PI will be equal to half the difference of the ſides AC, BC.

Cor. 2. If the great circles DR, EL, be drawn to make the ∠ ODR = ∠ OEL = $\frac{1}{2}$∠ ACB, and the ſides AC, CB be continued till they meet DR, EL in b, a; the triangles ACB, ADb, BEa will be ſimilar and equal in every reſpect.

Cor. 3. The circle deſcribed about the centre O, to touch the great circle AB, will alſo touch the great circles CAb, CBa, DbR, EaL.

Cor. 4. If PG, PH be each taken equal to AB, and the perpendicular arches GM, HN be drawn to meet DO, EO in M, N; M, N will be the centres, and (the equal arches) MG, HN the radii of the circles inſcribed in the triangles ADb, BEa.

Cor. 5. At I erect the perpendicular arch IS, to meet the great circle OQC in S; from S, O, demit upon CAb the perpendicular arches SK, OT; TK will be equal to half the baſe AB.

PROP.

PROP. XXVI. PROBLEM.

Given the angle which the arch bisecting the vertical angle makes with the base, and the sum of each side and its adjacent segment of the base made by the said bisecting arch, to construct the triangle.

Conſ. Let the great circles DABE, OQC, (fig. 87.) interſect each other in the given angle at Q; lay off QD, QE, equal to the given ſums reſpectively; biſect DE in P; let the arch PO. perpendicular to DE, meet the great circle OQC in O; with the centre O, and diſtance OD or OE, deſcribe the leſſer circle DCE, cutting the great circle OQC in C; join DC, EC; draw AC, BC, to make the angle DCA equal to the angle ADC, and the angle ECB equal to the angle BEC; ACB will be the triangle required. The *demonſtration* is evident from the laſt propoſition.

Remark. By help of the laſt propoſition the following *Problems* (as well as a great many others equally curious) may be eaſily conſtructed.

1. Given the vertical angle, perimeter and either the perpendicular or the arch biſecting the vertical angle.

2. Given the baſe, perpendicular and area.

3. Given the vertical angle and the ſum of each ſide and its adjacent ſegment of the baſe, made by the arch biſecting the ſaid angle.

4. Given the angle which the arch biſecting the vertical angle makes with the baſe, the ſum of the difference of the ſides and difference of the ſegments of the baſe made by the ſaid arch, and either the vertical angle, the arch biſecting the vertical angle, the perpendicular, the perimeter, the difference of the angles at the vertex made by the perpendicular, or the radius of the inſcribed circle.

PROP. XXVII. THEOREM.

om two given points on the ſame ſide of a
great circle, two arches be drawn to meet
make equal angles with the ſaid great circle;
e ſum of the arches ſo drawn will be leſs
ſum of any other two arches drawn from
two points to meet on the ſaid great circle.

t. Let A, B, (fig. 88.) be the two given
EDQ the given great circle, D the point
e arches AD, BD make the equal angles
BDQ; perpendicular to EDQ draw the
C to meet BD produced in C; draw the
Q, BQ, CQ to any other point Q in the
eat circle.

the right angled triangles ECD, EAD,
ng $\angle EDC (= \angle BDQ) = \angle ADE$,
id ED being common,
will be equal in every reſpect;
re $AE = EC$ and $AD = DC$;
in like manner $AQ = CQ$.
Trig. III. 13, CB is leſs than $BQ + CQ$:
$CB = AD + DB$,
id $BQ + CQ = AQ + BQ$;
$AD + BD$ is leſs than $AQ + BQ$;
e the ſum of the arches AD, BD is leſs
ſum of any other two arches AQ, BQ
om the ſame two points to meet on the
at circle.

Q. E. D.

f the leſſer circle HDI, touch the great
DQ in D, and fall on the contrary ſide
ne points A, B, and the arches AI, BI be
any other point I in the leſſer circle;
ſum of AD, DB is leſs than the ſum of

For, let BI cut the great circle EDQ in Q:
then AI + QI is greater than AQ:
therefore AI+QI+BQ is greater than AQ+BQ:
that is AI+BI is greater than AQ+BQ:
by this prop. AD+BD is less than AQ+BQ:
consequently AD+BD is less than AI + BI.

PROP. XXVIII. THEOREM.

If from three given points on the surface of the sphere, arches be drawn to a fourth point, such that their sum may be the least possible; I say, the position of that point must be such, that all the angles formed about it by those arches shall be equal among themselves.

Demon. Let A, B, C, (fig. 89.) be the three given points, and Q any other point where the angles AQC, BQC are unequal; about the pole C with the distance CQ, describe the lesser circle IDQH, and let D be that point in it where the angles ADC, BDC are equal; then I say the sum of AD, BD, CD is less than the sum of AQ, BQ, CQ.

For, the great circle EDF being described to touch the lesser one at D, the angles ADE, BDF will be equal;
therefore by prop. XXVII. AD+DB is less than AQ+BQ;
therefore AD+BD+CD is less than AQ+BQ+CQ;
therefore no point (Q) at which the angles are unequal can be the required one.

Q. E. D.

Cor. If the arches AD, BD, CD be produced till they meet on the opposite side of the sphere, they will make equal angles with each other at that point, and their sum will be the greatest possible.

PROP

PROP. XXIX. THEOREM. Fig. 90, Plate 6.

Of all ſpherical triangles ABC, ABQ having the ſame baſe, and the ſum of their other ſides the ſame, the iſoceles one ACB is the greateſt.

Demon. Let CPI, be drawn perpendicular to the baſe AB; let a leſſer circle QDR be drawn through Q perpendicular to CDPI, meeting it in D, and one (APB) equal and parallel thereto drawn through the points A, B; join AD, BD.

By prop. XXVII. *Cor.* AD + BD is leſs than AQ + BQ;
therefore AD+BD is leſs than AC+BC;
therefore the △ADB is leſs than the △ACB:
but the △ ADB = the △ ABQ,
For they are upon the ſame baſe, and between the ſame equal and parallel circles;
therefore the △ ABQ is leſs than the △ ACB.

Q. E. D.

Cor. 1. Of all ſpherical figures contained under the ſame perimeter, and number of ſides, the greateſt is when the ſides are all equal.

Cor. 2. Of all ſpherical triangles having the ſame perimeter, the equilateral one is the greateſt.

ARTICLE XXV.

Landen *on the Ellipſis and Hyperbola.*

[Continued from Page, 103.]

6. THE *whole fluent* of $\frac{1}{2} m^{\frac{1}{2}} z^{\frac{1}{2}} \dot{z} \div \sqrt{n^2 + 2fz - z^2}$, generated while z from o becomes $= m$, being equal to L; and the fluent of the ſame fluxion (ſuppoſing it to begin when z begins) being in general equal to L + AD — DP = FR — AF — dt; it appears that k being the value of z correſponding

refponding to the fluent L $+$ AD $-$ DP, $(mn^2 - n^2 k) \div (n^2 + mk)$ will be the value of z correfponding to the fluent L $+$ AF $-$ FR, and FR $-$ AE will be the part generated whilft z from $(mn^2 - n^2 k) \div (n^2 + mk)$ becomes $= m$. It follows, therefore, that the *tangent* dt, together with the fluent of $\frac{1}{2}m^{\frac{1}{2}}$ $\frac{1}{2}z^{\frac{1}{2}}\dot{z} \div \sqrt{n^2 + 2fz - z^2}$ generated whilft z from o becomes equal to any quantity k is equal to the fluent of the fame fluxion generated whilft z from $(mn^2 - n^2 k) \div (n^2 + mk)$ becomes $= m$; cp being taken $= n\sqrt{k \div m}$.

Suppofe $k = (m^2 - n^2 k) \div (n^2 + mk)$; its value will then be $n \cdot (m^2 + n^2)^{\frac{1}{2}} \div m - n^2 \div m$. Confequently the fluent of $\frac{1}{2}m^{\frac{1}{2}}z^{\frac{1}{2}}\dot{z} \div \sqrt{n^2 + 2fz - z^2}$ generated whilft z from o becomes $= n \cdot (m^2 + n^2)^{\frac{1}{2}} \div m - n^2 \div m$, together with the quantity $\sqrt{m^2 + n^2} - n$, is equal to the fluent of the fame fluxion generated whilft z from $n \cdot (m^2 + n^2)^{\frac{1}{2}} \div m - n^2 \div m$ becomes $= m$: and thefe two parts of the whole fluent being denoted by M and N refpectively, M will be $= n -$ AE, and N $= \sqrt{m^2 + n^2} -$ AE.

7. The fluent of $\frac{1}{2}m^{\frac{1}{2}}z^{\frac{1}{2}}\dot{z} \div \sqrt{n^2 + 2fz - z^2}$ being $=$ L $+$ AD $-$ DP, the fluent of $\frac{1}{2}m^{\frac{1}{2}}z^{\frac{1}{2}}\dot{z} \div \sqrt{n^2 + 2fz - z^2}$ $+$ DP $-$ AD $-$ L will be $=$ o. Therefore, the fluent of $\frac{1}{2}m^{\frac{1}{2}}z^{\frac{1}{2}}\dot{z} \div \sqrt{n^2 + 2fz - z^2}$ $+$ the fluent of $\frac{1}{2}m^{-\frac{1}{2}}n^2 z^{-\frac{1}{2}}\dot{z} \div \sqrt{n^2 + 2fz - z^2}$ being $=$ the fluent of $\frac{1}{2}z^{-\frac{1}{2}}\dot{z}\sqrt{(n^2 + mz) \div (m - z)}$, it is obvious that the fluent of $\frac{1}{2}m^{-\frac{1}{2}}n^2 z^{-\frac{1}{2}}\dot{z} \div \sqrt{n^2 + 2fz - z^2}$ is $=$ DP $-$ AD $-$ L $+$ the fluent of $\frac{1}{2}z^{-\frac{1}{2}}\dot{z}\sqrt{(n^2 + mz) \div (m - z)} =$ DP $-$

AD — L + the *elliptic arc* dg (Fig. 35, Plate 2.) whose abscissa cp is $= n\sqrt{z \div m}$. Consequently, putting E for $\frac{1}{4}$ of the periphery of that ellipsis, it appears that the *whole fluent* of $\frac{1}{2} m^{-\frac{1}{2}} n^2 z^{-\frac{1}{2}} \dot{z} \div \sqrt{n^2 + 2fz - z^2}$, generated whilst z from o becomes $= m$, is equal to $E - L = E + 2\,AE - n - \sqrt{m^2 + n^2}$.

8. By taking, in Art. 3. q, r, and s, each $= \frac{1}{2}$; and $a = -d = n^2 \div m$, $b = 1$, and $c = n^2$; we find, that, if y be $= (mn^2 - n^2 z) \div (n^2 + mz)$, $z^{-\frac{1}{2}} \dot{z} \div \sqrt{n^2 + 2fz - z^2} + y^{-\frac{1}{2}} \dot{y} \div \sqrt{n^2 + 2fy - y^2}$ will be $= 0$.

It is obvious therefore, that the fluent of $z^{-\frac{1}{2}} \dot{z} \div \sqrt{n^2 + 2fz - z^2}$, generated whilst z from o becomes equal to any quantity k, is equal to the fluent of the same fluxion, generated whilst z from $(mn^2 - n^2 k) \div (n^2 + mk)$ becomes $= m$.

Now, supposing $k = (mn^2 - n^2 k) \div (n^2 + mk)$, its value will be $n \cdot (m^2 + n^2)^{\frac{1}{2}} \div m - n^2 \div m$. Consequently the fluent of $z^{-\frac{1}{2}} \dot{z} \div \sqrt{n^2 + 2fz - z^2}$, generated whilst z from o becomes $= n \cdot (m^2 + n^2)^{\frac{1}{2}} \div m - n^2 \div m$, is equal to *half* the fluent of the same fluxion, generated whilst z from o becomes $= m$; which *half fluent* is known by the preceding article.

9. It appears by Article 4. that $\frac{1}{2} m^{\frac{1}{2}} y^{\frac{1}{2}} \dot{y} \div \sqrt{n^2 + 2fy - y^2} + \frac{1}{2} m^{\frac{1}{2}} z^{\frac{1}{2}} \dot{z} \div \sqrt{n^2 + 2fz - z^2}$ is $= -$ the *fluxion of the tangent* dt; and it appears by the last article, that $\frac{1}{2} m^{-\frac{1}{2}} n^2 y^{-\frac{1}{2}} \dot{y} \div$

$\sqrt{n^2 +}$

$\sqrt{n^2 + 2fy - y^2} + \frac{1}{2} m^{-\frac{1}{2}} n^2 z^{-\frac{1}{2}} \dot{z} \div \sqrt{n^2 + 2fz - z^2}$ is $= 0$; $mn^2 - n^2 y - n^2 z - myz$ being $= 0$. Therefore, by addition, we have $\frac{1}{2} y^{-\frac{1}{2}} \dot{y} \sqrt{(n^2 + my) \div (m - y)}$ $+ \frac{1}{2} z^{-\frac{1}{2}} \dot{z} \sqrt{(n^2 + mz) \div (m - z)} = -$ the *fluxion of the tangent* dt. Consequently, by taking the correct fluents, we find the *tangent* dt ($=$ the *tangent* fw) $=$ the *arc* ad — the *arc* fg! the abscissa cp being $= n\sqrt{z \div m}$, the abscissa cr $= n\sqrt{y \div m}$, and their relation expressed by the equation $n^6 - n^4 u^2 - n^4 v^2 - m^2 u^2 v^2 = 0$, u and v being put for $n\sqrt{z \div m}$ and $n\sqrt{y \div m}$ respectively. Moreover the tangents dt, fw, will each be $= m^2 uv \div n^3$, and ct $\times$ cw $=$ cv^2 $=$ ac $\times$ cg.

If for the semi-transverse axis cg we substitute h instead of $\sqrt{m^2 + n^2}$, the relation of u to v will be expressed by the equation $n^6 - n^4 u^2 - n^4 v^2 - (h^2 - n^2) \cdot u^2 v^2 = 0$, and dt ($=$ fw) will be $= (h^2 - n^2) \cdot uv \div n^3$.

If u and v be respectively put for fr and dp, their relation will be expressed by the equation $h^6 - h^4 u^2 - h^4 v^2 + (h^2 - n^2) \cdot u^2 v^2 = 0$, and dt ($=$ fw) will be $= (h^2 - n^2) \cdot uv \div h^3$.

10. Suppose y equal to z; (that is, $v = u$;) and that the points d and f coincide in e, in which case the tangent dt will be a *maximum*, and $=$ cg — ac. It appears then that the *arc* ae — the *arc* eg is $=$ cg — ac. Consequently, putting E for the quadrantal arc ag, we find that

the *arc* ae is $= \frac{E + h - n}{2}$!

the *arc* eg $= \frac{E - h + n}{2}$!

There

There are, I am aware, fome other parts of the arc ag whofe lengths may be affigned by means of the whole length (ag) with right lines; but to investigate fuch other parts is not to my prefent purpofe.

11. Taking m and n each $= 1$; that is ac ($=$ AC) $= 1$, and cg $= \sqrt{2}$; let the arc ag be then expreffed by e: put c for *one fourth* of the periphery of the circle whofe radius is 1; and let the *whole fluents* of $\frac{1}{2}z^{\frac{1}{2}}\dot{z} \div \sqrt{1-z^2}$ and $\frac{1}{2}z^{-\frac{1}{2}}\dot{z} \div \sqrt{1-z^2}$, generated whilft z from 0 becomes $= 1$, be denoted by F and G refpectively. Then, by what is faid above, $F + G = e$; and, by part X. of my *Mathematical Lucubrations*, it appears that $F \times G$ is $= \frac{1}{2}c$. From which equations we find $F = \frac{1}{2}e - \frac{1}{2}\sqrt{e^2 - 2c}$, and $G = \frac{1}{2}e + \frac{1}{2}\sqrt{e^2 - 2c}$.

But m and n being each $= 1$, L is $=$ F; therefore $1 + \sqrt{2} - 2$AE, the value of L from Art. 5. is, in this cafe, $= \frac{1}{2}e - \frac{1}{2}\sqrt{e^2 - 2c}$. Confequently, in the equilateral hyperbola, the *arc* AE, whofe abfciffa BC is $= \sqrt{1 + (1 \div \sqrt{2})}$, will be $= \frac{1}{2} + 1 \div \sqrt{2} - \frac{1}{4}e + \frac{1}{4}\sqrt{e^2 - 2c}$, by what is faid in the article laft mentioned. Hence the *rectification* of *that arc* may be effected by means of the *circle* and *ellipfis*!

ARTICLE.

ARTICLE XXVI.

Tables ... *heorems, for the calculation of Fluents, from* ... . Landen's *Memoirs, communicated by* Mr. ... liam Burdon.

TABLE III.

(*Continued from page 114.*)

XLII.

$\dot{y}$.
$-a^2)^{\frac{3}{2}}$

F ... e'')

$$= K + \frac{2}{a^2}.(de + DP - AD - L).$$

$$x = \frac{a\sqrt{y^2 - a^2}}{y}.$$

THEOREM XLIII.

The fluent of $\frac{y^{-\frac{1}{2}}\dot{y}}{(y^2-a^2)^{\frac{3}{4}}}$, generated whilst y from a becomes equal to $\sqrt{(1 \div \sqrt{2}) + \frac{1}{2}} \times a$, is $= \frac{M}{a^2}$.

THEOREM

THEOREM XLIV.

The *whole* fluent of $\dfrac{y^{-\frac{1}{2}}\dot{y}}{(y^2-a^2)^{\frac{3}{4}}}$ is $= \dfrac{2M}{a^2}$.

THEOREM XLV.

$$\dot{F} = \frac{\dot{y}}{(a^2+y^2)^{\frac{3}{4}}}.$$

$$F = K + \frac{4}{a^{\frac{3}{2}}}\cdot(æ + e'e'' - E''')$$

$$= K + \frac{2}{a^{\frac{3}{2}}}\cdot(æ + AD - DP).$$

$$x = a^2 \div \sqrt{a^2+y^2}.$$

THEOREM XLVI.

The fluent of $\dfrac{\dot{y}}{(a^2+y^2)^{\frac{3}{4}}}$, generated whilst y from o becomes equal to $\sqrt{2+\sqrt{2}} \times a$, is $= \dfrac{M}{a^{\frac{3}{2}}}$.

THEOREM XLVII.

The *whole* fluent of $\dfrac{\dot{y}}{(a^2+y^2)^{\frac{3}{4}}}$ is $= \dfrac{2M}{a^{\frac{3}{2}}}$.

THEOREM

ARTICLE XXVI.

Tables of Theorems, for the calculation of from Mr. Landen's *Memoirs, communic Mr.* William Burdon.

TABLE III.

(Continued from page 114.*)*

THEOREM XLII.

$$\dot{F} = \frac{y^{-\frac{1}{2}}\dot{y}}{(y^2-a^2)^{\frac{1}{2}}}.$$

$$F = K + \frac{4}{a^2}.(de - e'e'')$$

$$= K + \frac{2}{a^2}.(de + DP - AD -$$

$$x = \frac{a\sqrt{y^2-a^2}}{y}.$$

THEOREM XLIII.

The fluent of $\frac{y^{-\frac{1}{2}}\dot{y}}{(y^2-a^2)^{\frac{1}{2}}}$, generated whilst

a becomes equal to $\sqrt{(1 \div \sqrt{2}) + \frac{1}{2}} \times a$, is

THE

THEOREM XLIV.

The *whole* fluent of $\frac{y^{-\frac{1}{2}}\dot{y}}{(y^2-a^2)^{\frac{1}{2}}}$ is $=\frac{2M}{a^2}$.

THEOREM XLV.

$$\dot{F}=\frac{\dot{y}}{(a^2+y^2)^{\frac{3}{4}}}.$$

$$F=K+\frac{4}{a^{\frac{1}{2}}}\cdot(ae+e'e''-E''')$$

$$=K+\frac{2}{a^{\frac{1}{2}}}\cdot(ae+\hat{A}D-DP).$$

$$x=a^2\div\sqrt{a^2+y^2}.$$

THEOREM XLVI.

The fluent of $\frac{\dot{y}}{(a^2+y^2)^{\frac{3}{4}}}$, generated whilst y from o becomes equal to $\sqrt{2+\sqrt{2}}\times a$, is $=\frac{M}{a^{\frac{3}{2}}}$.

THEOREM XLVII.

The *whole* fluent of $\frac{\dot{y}}{(a^2+y^2)^{\frac{3}{4}}}$ is $=\frac{2M}{a^{\frac{3}{2}}}$.

THEOREM

THEOREM XLVIII.

$$\dot{F} = \frac{y^{-\frac{1}{2}} \dot{y}}{(a^2+y^2)^{\frac{3}{4}}}.$$

$$F = K + \frac{4}{a^2}, (de - e'e'')$$

$$= K + \frac{2}{a^2} (de + DP - AD - L).$$

$$e = ay \div \sqrt{a^2+y^2}.$$

THEOREM XLIX.

The fluent of $\frac{y^{-\frac{1}{2}} \dot{y}}{(a^2+y^2)^{\frac{3}{4}}}$, generated whilst y from o become equal to $\sqrt{(1 \div \sqrt{2}) - \frac{1}{2}} \times a$, is $= \frac{M}{a^2}$.

THEOREM L.

The *whole* fluent of $\frac{y^{-\frac{1}{2}} \dot{y}}{(a^2+y^2)^{\frac{3}{4}}}$ is $= \frac{2M}{a^2}$.

SCHEME

SCHEME *for* TABLE III.

$d\begin{cases} = \frac{1}{4} \text{ of the periphery of a circle whose radius is } 1. \\ = 1{\cdot}57079632. \end{cases}$

acd (fig. 26, plate 1.) is a quadrantal arc of an ellipsis $= E'$

Semi-transverse axis cd $= a\sqrt{2}$.

Semi-conjugate axis ac $= a$.

Abscissa cb $= 2^{\frac{1}{2}}\sqrt{a^2 - ax}$.

Ordinate be $= \sqrt{ax}$.

$e\begin{cases} = \text{the value of } E' \text{ when } a \text{ is } = 1. \\ = 1{\cdot}91009889. \end{cases}$

ae′e′d (fig. 27, pl. 1.) is a quadrantal arc of another ellipsis $= E'$

Semi-transverse axis $cd = \left(\frac{1}{\sqrt{2}} + \frac{1}{2}\right) \cdot a$.

Semi-conjugate axis $ac = \left(\frac{1}{\sqrt{2}} - \frac{1}{2}\right) \cdot a$.

$e'p'$ and its equal $e''p''$ (each $= \sqrt{a^2 - ax}$) are tangents, to which cp', cp'' are perpendiculars.

The abscissa cb', or cb'', corresponding to the ordinate $b'e'$ or $b''e''$, is $= \frac{2^{\frac{1}{2}}+1}{2^{\frac{7}{4}}} \times a^{\frac{1}{2}}\sqrt{2^{\frac{1}{2}}a + a - x \mp \sqrt{ax + x^2}}$.

$f\begin{cases} = \text{the value of } E'' \text{ when } a \text{ is } = 1. \\ = 1{\cdot}2545845059. \end{cases}$

AD

AD (fig. 28, pl. 1.) is an equilateral hyperbola, whose vertex is A and centre C.

DP is a tangent, to which CP is perpendicular.

$AC = a$, $CP = \sqrt{ax}$, $DP = \sqrt{a \div x} \times \sqrt{a^2 - x^2}$.

Abſciſſa CAB (correſponding to the ordinate BD)

$$= a \sqrt{(a + x) \div 2x}.$$

L, the *limit* of DP—AD, is $= 2E'' - E' = a \cdot (2f - e)$

$$= \tfrac{1}{2} a \cdot (e - \sqrt{e^2 - 2d}) = \cdot 5990701173 \times a.$$

$$M = 2(E' - E'') = 2a \cdot (e - f) = \tfrac{1}{2} a \cdot (e + \sqrt{e^2 - 2d})$$

$$= 1 \cdot 3110287771 \times a.$$

Note. All the Theorems in *Table III.* refer to this Scheme.

ARTICLE XXVII.

Of finding the Sums of certain Series by Mr. Stirling's *differential method, by Mr.* J. Mabbot, Mancheſter.

(Continued from page 109.*)*

18. REQUIRED the ſum of z terms of the ſeries $1^3 \cdot 1^2 + 2^3 \cdot 3^2 + 3^3 \cdot 5^2 + 4^3 \cdot 7^2 +$ &c.

Here $T = z^3 \cdot (1 + 4z^2 - 4z) = z^3 + 4z^5 - 4z^4$ $= z + 35z \cdot \overline{z-1} + 77z \cdot \overline{z-1} \cdot \overline{z-2} + 36z \cdot \overline{z-1} \cdot \overline{z-2} \cdot \overline{z-3} + 4z \cdot \overline{z-1} \cdot \overline{z-2} \cdot \overline{z-3} \cdot \overline{z-4}$, the values of z being 1, 2, 3, &c.

$$\text{and } S = \overline{z+1} \times \left(\frac{8z - 13z^2 - 37z^3 + 32z^4 + 40z^5}{60}\right)$$

$$= \frac{2}{3} z^6 + \frac{6}{5} z^5 - \frac{1}{12} z^4 - \frac{5}{6} z^3 - \frac{1}{12} z^2 + \frac{2}{15} z.$$

19.

19. Required the ſum of z terms of the ſeries $1\cdot2^2\cdot3^3 + 3\cdot4^2\cdot5^3 + 5\cdot6^2\cdot7^3 +$ &c. Here T $=(2z-1)\cdot(2z)^2\cdot(2z+1)^3 = 64z^6+64z^5-16z^3-4z^2$ $=108z+2892z\cdot\overline{z-1}+7344z\cdot\overline{z-1}\cdot\overline{z-2}+4800z\cdot\overline{z-1}\cdot\overline{z-2}\cdot\overline{z-3}+1024z\cdot\overline{z-1}\cdot\overline{z-2}\cdot\overline{z-3}\cdot\overline{z-4}+64z\cdot\overline{z-1}\cdot\overline{z-2}\cdot\overline{z-3}\cdot\overline{z-4}\cdot\overline{z-5}$,
the values of z being 1, 2, 3, 4, &c.

and $S = \overline{z+1} \times \left(\frac{192z^6+704z^5+640z^4-164z^3-256z^2+18z}{21}\right)$

$$= \frac{64}{7}z^7 + \frac{128}{3}z^6 + 64z^5 + \frac{68}{3}z^4 - 20z^3 - \frac{34}{3}z^2 + \frac{6}{7}z.$$

20. Required the ſum of any number of terms (z) of the ſeries $1\cdot2^2+2\cdot3^2+3\cdot4^2+4\cdot5^2+$&c.
Here T $= z\cdot(z+1)^2 = 4z+5z\cdot\overline{z-1}+z\cdot\overline{z-1}\cdot\overline{z-2}$,
the values of z being 1, 2, 3, 4, &c.

and $S = \overline{z+1} \times \left(2z + \frac{5}{3}z\cdot\overline{z-1} + \frac{1}{4}z\cdot\overline{z-1}\cdot\overline{z-2}\right)$

$$= \frac{1}{4}z^4 + \frac{7}{6}z^3 + \frac{7}{4}z^2 + \frac{5}{6}z.$$

21. Required the ſum of the infinite ſeries?

$$\frac{1}{2\cdot3\cdot4\cdot5} + \frac{1}{3\cdot4\cdot5\cdot6} + \frac{1}{4\cdot5\cdot6\cdot7} + \text{\&c.}$$

Here T $= \frac{1}{z\cdot\overline{z+1}\cdot\overline{z+2}\cdot\overline{z+3}}$; the values of z being 1, 2, 3, &c.

and $S = \frac{1}{3z\cdot\overline{z+1}\cdot\overline{z+2}} = \frac{1}{72}$, when z is taken $= 1$.

22. Suppoſe, the ſeries to be ſummed be

$$\frac{3}{4\cdot10\cdot12} + \frac{4}{5\cdot12\cdot14} + \frac{5}{6\cdot14\cdot16} + \text{\&c.}$$

Here $T = \frac{z}{z+1\cdot 2z+4\cdot 2z+6}$

$= \frac{1}{4z\cdot z+1} - \frac{5}{4z\cdot z+1\cdot z+2} + \frac{9}{4z\cdot z+1\cdot z+2\cdot z+3}$,

the values of z being 3, 4, 5, &c.

and $S = \frac{1}{4z} - \frac{5}{8z\cdot z+1} + \frac{3}{4z\cdot z+1\cdot z+2}$

$= \frac{2z^2+z}{8z^2 z+1\cdot z+2} = \frac{2z+1}{8\cdot z+1\cdot z+2} = \frac{7}{160}$, when z is taken $= 3$.

23. The ſeries to be ſummed being

$$\frac{1}{2\cdot 4} + \frac{1}{4\cdot 6} + \frac{1}{6\cdot 8} + \frac{1}{8\cdot 10} + \text{\&c.}$$

Here $T = \frac{1}{2z\cdot 2z+2} = \frac{1}{4z\cdot z+1}$; the values of z being 1, 2, 3, &c.

and $S = \frac{1}{4z} = \frac{1}{4}$, when z is taken $= 1$.

24. What is the ſum of the ſeries?

$$\frac{6}{2\cdot 4\cdot 6} + \frac{6}{4\cdot 6\cdot 8} + \frac{6}{6\cdot 8\cdot 10} + \text{\&c.}$$

Here $T = \frac{6}{2z\cdot 2z+2\cdot 2z+4} = \frac{6}{8z\cdot z+1\cdot z+2}$,

the values of z being 1, 2, 3, &c.

and $S = \frac{6}{16z\cdot z+1} = \frac{6}{32} = \frac{3}{16}$, when z is taken $=1$.

25. Find the ſum of the ſeries

$$\frac{2}{4\cdot 6\cdot 8} + \frac{4}{6\cdot 8\cdot 10} + \frac{6}{8\cdot 10\cdot 12} + \text{\&c.}$$

Here $T = \frac{2z}{2z+2\cdot 2z+4\cdot 2z+6} = \frac{2z}{6\cdot z+1\cdot z+2\cdot z+3}$

=

$$=\frac{1}{4z\cdot\overline{z+1}}-\frac{5}{4z\cdot\overline{z+1}\cdot\overline{z+2}}+\frac{9}{4z\cdot\overline{z+1}\cdot\overline{z+2}\cdot\overline{z+3}};$$

the values of z being 1, 2, 3, &c. and S $=$

$$\frac{1}{4z}-\frac{5}{8z\cdot\overline{z+1}}+\frac{3}{4z\cdot\overline{z+1}\cdot\overline{z+2}}=\frac{2z+1}{8z\cdot\overline{z+1}\cdot\overline{z+2}}=\frac{1}{16},\text{ when } z=1.$$

26. Let the ſeries to be ſummed be

$$\frac{1}{2\cdot3\cdot4\cdot5}+\frac{4}{3\cdot4\cdot5\cdot6}+\frac{9}{4\cdot5\cdot6\cdot7}+\&c.$$

Here T $=\frac{z^2}{\overline{z+1}\cdot\overline{z+2}\cdot\overline{z+3}\cdot\overline{z+4}}=\frac{1}{z\cdot\overline{z+1}}-\frac{9}{z\cdot\overline{z+1}\cdot\overline{z+2}}$

$$+\frac{37}{z\cdot\overline{z+1}\cdot\overline{z+2}\cdot\overline{z+3}}-\frac{64}{z\cdot\overline{z+1}\cdot\overline{z+2}\cdot\overline{z+3}\cdot\overline{z+4}};$$

the values of z being 1, 2, 3, &c. and S $=$

$$\frac{1}{z}-\frac{9}{2z\cdot\overline{z+1}}+\frac{37}{3z\cdot\overline{z+1}\cdot\overline{z+2}}-\frac{64}{4z\cdot\overline{z+1}\cdot\overline{z+2}\cdot\overline{z+3}}$$

$$=\frac{6z^2+9z+5}{6z\cdot\overline{z+1}\cdot\overline{z+2}\cdot\overline{z+3}}=\frac{5}{36}\text{ when } z \text{ is taken} = 1.$$

27. Let the ſeries propoſed be

$$\frac{1}{3\cdot5\cdot7}+\frac{2}{4\cdot6\cdot8}+\frac{3}{5\cdot7\cdot9}+\frac{4}{6\cdot8\cdot10}+\&c.$$

Here T $=\frac{z}{\overline{z+2}\cdot\overline{z+4}\cdot\overline{z+6}}=\frac{1}{z\cdot\overline{z+1}}-\frac{11}{z\cdot\overline{z+1}\cdot\overline{z+2}}$

$$+\frac{66}{z\cdot\overline{z+1}\cdot\overline{z+2}\cdot\overline{z+3}}-\frac{246}{z\cdot\overline{z+1}\cdot\overline{z+2}\cdot\overline{z+3}\cdot\overline{z+4}}$$

$$+\frac{540}{z\cdot\overline{z+1}\ldots\ldots\ldots\ldots\overline{z+5}}-\frac{540}{z\cdot\overline{z+1}\cdot\overline{z+2}\ldots\ldots\overline{z+6}}$$

the values of z being 1, 2, 3, &c. and S $=$

$$\frac{1}{z}-\frac{11}{2z\cdot\overline{z+1}}+\frac{22}{z\cdot\overline{z+1}\cdot\overline{z+2}}-\frac{123}{2z\cdot\overline{z+1}\cdot\overline{z+1}\cdot\overline{z+3}}$$

$$+\frac{108}{z\cdot\overline{z+1}\,.\,.\,.\,.\,.\,.\,.\,\overline{z+4}}-\frac{90}{z\cdot\overline{z+1}\cdot\overline{z+2}\,.\,.\,.\,.\,.\,.\,.\,.\,.\,\overline{z+5}}$$

$$=\frac{2z^5+19z^4+60z^3+74z^2+31z}{2z\cdot\overline{z+1}\cdot\overline{z+2}\cdot\overline{z+3}\cdot\overline{z+4}\cdot\overline{z+5}}=\frac{186}{1440}=\frac{31}{240},\text{ when } z=1.$$

ARTICLE XXVIII.

GENERAL PROBLEMS.

BY MR. JOHN LOWRY.

PROBLEM I. Fig. 101, 102, 103, 104. *Plate* 7, 8.

LET there be any number m of given points A, B, C, &c. and let P be any other given point; through P it is required to draw a right line, ſuch that the ſum of the perpendiculars AX, BY, CZ, &c. falling thereon from the given points A, B, C, &c. may be equal to a given magnitude S.

Firſt. Let $m = 2$.

Caſe I. When the given points are on the ſame ſide of the required line (fig. 101).

ANALYSIS.

Suppoſe the problem ſolved, and that PXY is really the line to be drawn.

Join AB and biſect it in b; draw bP and on it produced take PV equal to twice bP, and demit the perpendiculars VR, bz.

By hypotheſis AX + BY = S:
but AB is biſected in b;
therefore AX + BY = 2bz
wherefore 2bz = S;
now, the right angled triangles VRP, bzP,
having ∠ RPV = ∠ b P z and VP = 2 bP,
will alſo have RV = 2bz;
therefore RV = S,
that is, VR is equal to a given magnitude,
and VP is given in a magnitude and poſition;
wherefore PXY is given in poſition.

SYNTHESIS.

Conſ. Join AB and biſect it in b; join bP and produce it till PV is equal to twice bP; then on PV as a diameter deſcribe a circle, and in it apply VR equal

equal to S; through the points P, R draw a right line and it will be that required.

Demon. From b demit the perpendicular bz.

By *conf.* VR = S:

but the right angled triangles VRP, bzP are similar, having the ∠VPR = ∠bPz and VP = 2bP,

and therefore VR = 2bz;

wherefore 2bz = S:

but AB is bisected in b;

and therefore AX + BY = 2bz;

wherefore AX + BY = S.

Q. E. D.

Observation. Since two equal right lines VR, Vr may be inscribed in the circle, it is evident that there may be two positions of the required line, except when VR coincides with VP; and then the sum of the perpendiculars will be a *maximum.*

Case II. When the given points are on different sides of the required line (fig. 102).

ANALYSIS.

Imagine the thing to be effected, and that PXY is really the line required.

Produce the perpendicular AX till XR be equal to BY; join BR, BA.

By hypothesis AX + BY = S,

or AX + XR = S,

or AR = S,

that is, AR is equal to a given magnitude:

but AB is given in magnitude and position;

therefore BR is given in position;

wherefore PXY will be given in position, because it passes through the given point P and is parallel to BR.

SYNTHESIS.

Conſ. Join the given points A, B, and on it as a diameter deſcribe a circle, in which apply AR = to S and join BR; through P draw the right line PXY parallel to BR, and it will be the line required.

The demonſtration is evident from the analyſis.

Obſervation. Since two equal right lines AR, Ar may be inſcribed in the circle, it is evident that there may be two poſitions of the required line, except when AR coincides with AB, in which caſe the ſum of the perpendiculars will be a *maximum*.

Second. Suppoſe $m = 3$.

Caſe III. When the given points are all on the ſame ſide of the required line (fig. 103).

ANALYSIS.

Conceive the thing done, and PYXZ the line required.

Find the point V as in *Caſe I.* and join VC; demit the perpendiculars CZ, VR; produce VR till RT is equal to CZ; join CT.

By hypotheſis $AX + BY + CZ = S$;
by *Caſe I.* $AX + BY = VR$;
but $CZ = RT$;
therefore $VR + RT = S$;
or $VT = S$;

that is, VT is equal to a given magnitude, and VC is given in magnitude and poſition; therefore CT is given in poſition; wherefore PYXZ, which is parallel to CT and paſſes through a given point P, will be given in poſition.

SYNTHESIS.

Conſ. Find the point V as in *Caſe I.* and join VC, upon which as a diameter deſcribe a circle,

and

and in it apply VT equal to S; join CT and parallel thereto, through P draw the right line PYXZ, and it will be that required.

The demonſtration is evident from the analyſis.

Obſervation. The ſum of the perpendiculars will be a *maximum* when VT coincides with VC.

Caſe IV. When the given points are not all on the ſame ſide of the line required (fig. 104).

ANALYSIS.

Let us conceive the thing effected, and that PXYZ is really the line ſought.

Find the point V as in *Caſe I.* join VC and biſect it in W; draw WP and produce it till PS be equal to twice WP; drop the perpendiculars ST, VR.

By hypotheſis	$AX + BY + CZ = S$,
and by *Caſe I.*	$AX + BY = VR$;
therefore	$CZ + VR = S$:
but	VC is biſected in W;
therefore	$CZ + VR = 2WP = ST$;
wherefore	$ST = S$,

that is, ST is equal to a given magnitude, and SP is given in magnitude and poſition; therefore TPXYZ is given in poſition.

SYNTHESIS.

Conſ. Find the point V as in *Caſe I.* and join VC and biſect it in W; draw WP and produce it till PS be equal to twice WP; upon PS as a diameter let a circle be deſcribed, and in it apply ST equal to S: through the points T, P draw a right line, and it will be the one required.

The

The demonstration of this is also evident from the analysis.

Observation 1. The sum of the perpendiculars will be a *maximum* when ST coincides with PS.

Observation 2. By the same *simple analysis and synthesis*, the position of the required line may easily be determined, as well when the lines drawn from the given points make given angles with the required line, as when they are perpendicular thereto.

PROBLEM II. Fig. 105, 106. *Plate* 8.

Let there be any number of given points A, B, C, &c. and likewise a circle given in magnitude and position; it is required to draw a right line to touch the circle, such that the sum of the perpendiculars AX, BY, CZ, &c. falling thereon from the given points A, B, C, &c. may be equal to a given line.

Case I. When the given points are all on the same side of the required line. (Fig. 105).

ANALYSIS.

Suppose the problem solved, and that XZRY touching the circle in R is really the line required to be drawn.

Let P be the centre of the given circle and join PR, which will be perpendicular to XZRY; through P, parallel to XZRY draw xzPy, meeting the perpendiculars AX, BY, CZ, &c. in x, y, z, &c.

Then, since the sum of the perpendiculars AX, BY, CZ, &c. is given, and the radius PR (= Xx = Yy = Zz = &c.) being also given; the sum of the perpendiculars Ax, By, Cz, &c. will be given, being equal to the difference between the sum of the perpendiculars AX, BY, CZ, &c. and the multiple by the number of the given points A, B, C, &c. of the radius of the given circle: therefore the problem is reduced

reduced to the ſame as the laſt, *viz.* to draw a right line through the given point P, ſuch that the ſum of the perpendiculars Ax, By, Cz, &c. falling thereon from the given points A, B, C, &c. may be equal to a given line.

Obſervation. The *maximum* or *limits* is alſo eaſily determined, as in the laſt problem.

Caſe II. When the given points are not all on the ſame ſide of the line required. (fig. 106).

ANALYSIS.

By arguments ſimilar to thoſe uſed in the preceding caſe, it appears that the ſum of the perpendiculars Ax, By, Cz, &c. is equal to a given quantity, being equal to the ſum of the perpendiculars AX, BY, CZ, &c. made leſs by the multiple of the radius of the given circle, by the difference of the numbers of the given points on the reſpective ſides of the required line; therefore the problem in this caſe is alſo reduced to the ſame as the laſt problem; whence the *conſtruction* and *limits* are eaſily determined, and that not only when the lines drawn from the given points are perpendicular to, but alſo when they make any given angle with the required line.

(To be continued.)

ARTIC[illegible]

ARTICLE XXIX.

Demonstrations to Lawſon's *Propoſitions propoſed in* ARTICLE XVIII. *and likewiſe Demonſtrations to Propoſitions III. and IV. which were propoſed in No. I.*

We may alſo obſerve in this place that queſtions 1ſt and 2nd of this Work, are Propoſitions I. and II. of Lawſon's Collection.

PROP. III. (queſt. 17).

Demonſtrated by Peletarius.

ANALYSIS. Fig. 107, 108. *Plate* 8.

SINCE GD : DF :: GH : HF;
by alternation GD : GH :: DF : HF:
but DG : GH :: DC : AH,
and DF : FH :: BD : AH;
wherefore DC : AH :: BD : AH;
therefore DC = BD;
wherefore BC is biſected in D.

Q. Q. V.

SYNTHESIS.

Becauſe BC is biſected in D,
we have DC = CB;
therefore DC : AH :: BD : AH:
but DC : AH :: DG : GH;
and BD : AH :: DF : FH,
wherefore DG : GH :: DF : FH,
and by alternation DG : DF :: GH : FH.

Q. E. D.

The ſame by Mr. Lowry, *Solihull.*

By ſim. △'s DC : GD :: AH : GH,
and DF : BD :: FH : AH;
hence, by equality DG : DF :: GH : FH.

Q. E. D.

The

The converſe of this is alſo true.

Let there be a triangle ABC, and through the vertex A a line AE drawn parallel to the baſe BC, and any line drawn through a point D in BC, to meet AB, AC, AE in F, G, H, ſuch that GD : DF :: GH : HF; then I ſay BC is biſected in D.

By hypotheſis	GD : DF :: GH : HF,
and by ſim. $\triangle$'s	DC : GD :: AH : GH;
therefore by equality	DF : CD :: HF : AH.
Again, by ſim. $\triangle$'s	DF : BD :: HF : AH;
hence, by equality	DC : BD :: AH : AH;
therefore	DC = BD.

Q. E. D.

The ſame by Mr. I. H. Swale, *Leeds.*

Join BG and let it meet AE in I.

By ſim. $\triangle$'s we have	BF : FA :: FD : FH,
and	BF : FA :: BD : AH;
therefore	BD : AH :: FD : FH :
but	BD : HI :: GD : GH;
therefore	GD : GH :: FD : FH;
or	GD : DF :: GH : HF.

Q. E. D.

The ſame by Mr. Campbell, *Kendal.*

By ſim. $\triangle$'s	BD : FD :: AH : HF,
and	DC or BD : DG :: AH : GH;
wherefore	DG : FD :: GH : HF.

Q. E. D.

PROP. IV.

Demonſtrated by Peletarius.

ANALYSIS, Fig. 109, 110. *Plate* 8.

Draw CG perpendicular to AC meeting EF in G; and draw EH, FK perpendiculars to AB meeting it in H,

H, K, alſo let EL, FM be drawn parallel to DC meeting CG in L, M.

By hypotheſis EC : CF :: ED : DF:
but ED : DF :: EH : FK;
therefore EC : CF :: EH : FK:
but CHE, CKF are right-angles;
therefore the △'s CHE, CKF are equi-angular,
wherefore CH : CK :: EH : FK:
but CH : CK :: EL : FM :: EG : GF,
and EH : FK :: ED : DF;
therefore EG : GF :: ED : DF,
which is true by the ſecond propoſition.

SYNTHESIS.

By Prop. II. EG : GF :: ED : DF:
but EG : GF :: EL : FM :: HC : CK,
and ED : DF :: EH : FK;
therefore CH : CK :: EH : FK;
therefore the △'s CHE, CKF are equi-angular,
wherefore EC : CF :: EH : FK:
but EH : FK :: ED : DF;
therefore EC : CF :: ED : DF.

Q. E. D.

The ſame by Mr. Lowry.

Draw CG perpendicular AB, and let it meet EF in G; through G draw RGQ parallel to AB, meeting EC, CF in R, Q.

Now, by Prop. II. ED : DF :: EG : GF;
therefore by conv. Prop. III. GQ=GR;
wherf. the rt. ∠'d △'s RGC, QGC, are=in all reſpects;
therefore ∠GCQ=∠GCR and ∠GQC=∠GRC;
but ∠GQC=∠BCF and ∠GRC=∠DCE;
wherefore ∠BCF = ∠DCE.

Hence

Hence, Eu. VI. 3. (in fig. 110.) EC : CF :: EG : GT :
but EG : GF :: ED : DT ;
therefore EC : CF :: ED : DF.
gain, Eu. VI. 3. (in fig. 109.) EC : CE :: ED : DF.

Q. E. D.

The converſe of this is alſo true.

f in AB, the diameter of a circle, a point D be
n, and DEF be drawn to meet the circle in
nd F; and EC, FC, be drawn to make equal
es with the diameter; then, I ſay, AC : CB :: AD
3.

et the lines be drawn as in the propoſition.
By hypotheſis ∠DCE = ∠BCF :
it ∠DCE = ∠GRC and ∠BCF = ∠GQC ;
therefore ∠GQC = ∠GRC ;
rf. the rt. ∠'d △'s RGC, QGC are = in all reſpects;
and therefore GQ = GR ;
e by Prop. III. ED : DF :: EG : GF ;
therf. by conv. Prop. II. AC : CB :: AD : DB.

Q. E. D.

The ſame by Mr. Swale.

ANALYSIS.

EA, being joined, will biſect the ∠DEC.
By hypotheſis AC : AD :: CB : BD :
Eu. VI. 3. AC : AD :: CE : DE ;
herefore BD : DE :: CB : CE :
, by the circle DB : DE :: DF : DA ;
therefore DF : DA :: CB : CE :
but CE : CB :: CA : CF ;
refore CA : CF :: DA : DF,
or CA : DA :: CF : DF.

Q. Q. T.

T SYN-

SYNTHESIS.

By analysis	CA : DA :: CF : DF;
and	CA : DA :: CE : DE;
therefore	EC : CF :: DE : DF.

Q. E. D.

The same by Mr. Campbell.

From O, the centre of the circle, draw OE, OF, and produce FC to form the external angle ECN of the triangle CEF.

By hypothesis	AC	:	CB	::	AD	: DB.
Comp. (fig. 109.)	AC+CB	:	CB	::	AD+DB	: DB,
that is	2CO	:	CB	::	2OE	: DB,
or	CO	:	CO+OE	::	OE	: DO+OE;
invertendo	CO+OE	:	CO	::	DO+OE	: OE.
Again, inver. (fig. 110.)	CB	:	AC	::	DB	: AD.
convertendo	CB	:	CB—AC	::	DB	: DB—AD
that is	CO+OE	:	2CO	::	DO+OE	: 2OE,
or	CO+OE	:	CO	::	DO+OE	: OE.
Hence, div. (fig. 109, 110.)	OE or OF	:	CO	::	DO	: OE or OF;

wheref. the triangles COE, DOE are equi-angular, and the triangles COF, DOF are also equi-angular; theref. Eu. VI. 6. $\angle$OCE=$\angle$DEO & $\angle$OCF=$\angle$DFO hence, $\angle$DCE=$\angle$OEF=$\angle$DFO=$\angle$OCF=$\angle$NCA; wheref. Sim. Eu. VI. Prop. A. EC : CF :: ED : DF.

Q. E. D.

PROP. V.

Demonstrated by Peletarius.

ANALYSIS. *Fig.* 111, 112. *Plate* 8.

Let FK, GL parallel to DE meet AE, in K, L, and GM, FN parallel to AE meet DE in M, N; and also let FG meet AE in H.

By

By hypothesis EF : EG :: FD : DG :
but FD : DG :: FN : GM ;
therefore EF : EG :: FN : GM ;
and therefore the △'s ENF, EMG are equi-angular,
therefore EN : EM :: FN : GM :
but EN : EM :: FK : GL :: FH : HG,
and FN : GM :: DF : DG ;
therefore FH : HG :: DF : DG,
which is true by the third proposition.

SYNTHESIS.

By Prop. III. FH : HG :: DF : DG :
but FH : HG :: FK : GL :: EN : EM,
and DF : DG :: FN : GM ;
therefore EN : EM :: FN : GM ;
wherefore the △'s ENF, EMG are equi-angular ;
therefore EF : EG :: FN : GM :
but FN : GM :: DF : DG ;
therefore EF : EG :: DF : DG.
Q. E. D.

The same by Mr. Lowry.

By Prop. III. GD : DF :: GH : HF :
but EH is parallel to QP ;
therefore, conv. Prop. III. QD = DP ;
and therefore the triangle QEP is isoceles ;
wherefore ED bisects the ∠FEG (QEP) ;
therefore Eu. VI. 3. EF : EG :: DF : DG.
Q. E. D.

The same by Mr. Swale.

ANALYSIS.

Join BG ; produce AE to meet BG, GD in I, H.
By parallels GD : GH :: DB (DC) : HI (HA),

and FD : FH :: DB : HI;
therefore GD : GH :: FD : FH.

Q. Q. V.

SYNTHESIS.

By analysis GH : HF :: GD : FD;
by reason of = ∠'s GH : HF :: GE : EF;
hence, by equality EF : EG :: FD : DG.

Q. E. D.

The same by Mr. Campbell.

Let FG meet AE in H; draw GMS parallel to BC, meeting FE, DE in S, M; produce FE to R making ER equal to ES, and join RG.

Then, because of the parallels EH, SG,
we shall have SE : FE :: GH : FH,
and, by Prop. III. GD : DF :: GH : FH;
therefore SE or ER : FE :: DG : FD;
wherefore DE is parallel to GR;
and, since SE = ER, SM will be = MG:
but DE being perpendicular to BC, is also ⊥ to SG; hence, it appears that the triangle SEG is isosceles, having its vertical angle SEG bisected by DE; wheref. Sim. Eu. VI. 3 or A. FE : EG :: FD : DG.

Q. E. D.

PROP. VI.

Demonstrated by Peletarius.

ANALYSIS. Fig. 113, 114. *Plate* 8.

By hypothesis GD : DH :: GC : CH:
but GC : CH :: AG : BH;
therefore GD : DH :: AG : BH;
and therefore the △'s GAD, BDH are equi-angular; therefore

therefore GA : BH :: AD : BD :
but GA : BH :: AC : CB ;
therefore AC : CB :: AD : DB.

Q. Q. V.

SYNTHESIS.

Becaufe AC : CB :: AD : DB,
and GA : BH :: AC : CB ;
it will be GA : BH :: AD : DB ;
therefore the △'s GAD, BDH are equi-angular ;
and therefore GD : DH : AG : BH :
but GA : BH :: GC : CH ;
therefore GD : DH :: GC : CH.

Q. E. D.

The demonſtration by Mr. Lowry *is exactly the ſame as the* Syntheſis *by* Peletarius.

The ſame by Mr. Swale.

ANALYSIS.

Let DG meet FB in I ; produce GA to meet DH in T, and draw TI which will paſs through C.

Then, GA = AT, IB = BH, CG = CT, CI = CH, &c.
and by parallels CA : CB :: CG : CH :: AD : DB.

Q. Q. V.

SYNTHESIS.

By analyſis AD : DB :: CG : CH ;
by ſim. △'s AD : DB :: DG or DT : DI or DH ;
therefore by equality DG : DH :: GC : CH.

Q. E. D.

The ſame by Mr. Campbell.

Let BF meet GD in I, and join IC,
By hypotheſis AC : CB :: AD : DB ;
by ſim. △'s AC : CB :: AG : BH,

and AD : DB :: AG : BI;
therefore AG : BH :: AG . BI;
and therefore BH is equal to BI,
wherefore, since HI is perpendicular to DC,
the △'s IHD, IHC are evidently isosceles;
therefore DH = DI and IC = CH;
wherefore in fig. 113. DC bisects the ∠GDH;
and in fig. 114. DC bisects the ∠GCI;
theref. Eu. VI. 3 (fig. 113, 114.) GD: DI or DH :: GC : CI or CH.

Q. E. D.

PROP. VII.

Demonstrated by Peletarius.

ANALYSIS. Fig. 115, 116. *Plate* 8.

Let K be the centre of the circle; join GK and produce it to meet the circle in L; join DL.

By hypothesis AC : CB :: AH : BH,
and AB is bisected in K;
theref. Prop. I. rect. CHK = rect. AHB = rect. GHE;
wheref. the points C, E, K, G are in a circle;
therefore the ∠HKG = the ∠DEG = the ∠DLG;
therefore DL, AK are parallel;
wherefore the ∠GFK = the ∠GDL;
but the ∠GDL is a right angle;
therefore the ∠GFK is a right angle;
and therefore DF is perpendicular to AB.

Q. Q. V.

SYNTHESIS.

Because DF is perpendicular to AB,
the ∠GFK will be a right angle:
but the ∠GDL is a right angle;
therefore AK, DL are parallel;
wherefore the ∠HKG = the ∠DLG = the ∠DEG;

therefore

therefore the points C, E, K, G are in a circle;
therefore rect. CHK = rect. GHE = rect. AHB:
but AB is bisected in K and in it two points C, H are found such, that rect. CHK = rect. AHB;
therefore conv. Prop. I. AC : CB :: AH : HB.

Q. E. D.

Conversely.

ANALYSIS.

Let K be the centre of the circle; join GK and produce it to meet the circle in L; join DL and let DG meet AB in F.

Since DG is perpendicular to AB,
the ∠GFK will be a right angle:
but the ∠GDL is a right angle;
therefore AK, DL are parallel;
therefore the ∠GKH = the ∠GLD = the ∠GED;
wherefore the points C, E, K, G are in a circle;
therefore rect. CHK = rect. EHG = rect. AHB:
but AB is bisected in K and in it two points C, H are found such, that rect. CHK = rect. AHB;
therefore conv. Prop. I. AC : CB :: AH : HB.

Q. Q. V.

SYNTHESIS.

Since AC : CB :: AH : HB,
and AB is bisected in K;
by Prop. I. rect. CHK = rect. AHB = rect. EHG;
therefore the points C, E, K, G are in a circle;
therefore the ∠GKH = the ∠GED = the ∠GLD;
wherefore AK, DL, are parallel;
therefore the ∠GFK = the ∠GDL:
but the ∠GDL is a right angle;
therefore the ∠GFK is a right angle;
wherefore DG will be perpendicular to AB.

Q. E. D.

The

The ſame by Mr. Lowry.

Join DH, DC.
Then the right angled triangles GFH, DFH; having DF = FG and FH common, will alſo have ∠DHF = ∠GHF: but ∠GHF = ∠EHB; theref. ∠DHF = ∠EHB; wherefore conv. Prop. IV. AC : CB :: AH : HB.

Q. E. D.

Converſely.

By hypotheſis AC : CB :: AH : HB; theref. by my demon. to Prop. IV. ∠DHF = ∠EHB (GHF). Now, the △'s GFH, DFH, having FH common, alſo DH = HG and ∠GHF = ∠DHF, will likewiſe have DF = FG; theref. Eu. III. 3. DG will be perpendicular to AB.

Q. E. D.

Cor. Produce GC to meet the circle again in Q; then if QE be joined, it will be perpendicular to AB.

The ſame by Mr. Swale.

ANALYSIS.

Produce GC to meet the circle in Q.

Becauſe	DG cuts AB at right angles;
we ſhall have	DF = FG and GC = CD:
but by Prop. IV.	CG : CE :: GH : HE;
and	GH : CG :: HA : AC;
therefore	HA : AC :: HE : CE,
or	CA : CE :: HA : HE.

Q. Q. V.

SYNTHESIS.

By analyſis	CA · HE = CE · HA,
and by the circle	CA · CB = CE · CG;

therefore

therefore	HA : HE :: CG : CB;
again, by the circle	HA · HB = HG · HE;
or	HA : HE :: HG : HB;
therefore	CG : HG :: CB : HB:
but	CG : HG :: AC : AH;
wherefore	AC : CB :: AH : HB.

Q. E. D.

Converſely.

In this caſe the triangles GCF, DCF; having GC = CD, the ∠GCF = the ∠DCF and FC common, will alſo have GF = FD; therefore Eu. III. 3. DG will be perpendicular to AB.

Q. E. D.

The ſame by Mr. Campbell.

Draw INCM perpendicular to CB; produce DH to meet the right line INCM in I and the circle in Q; join EQ and let it meet CB in O; draw DN, MQ parallel to CB.

Then by Eu. III. 3.	DG is biſected in F;

therefore in the triangles DFH, GFH, we have DF = FG, ∠DFH = ∠GFH and FH common; wherefore DH = HG and ∠DHF = ∠FHG;

therefore Sim. Eu. VI. A or 3.	CD : CE :: DH : HE or HQ:
but by ſim. △'s	CD:CE::CF : CO,
that is	CD:CE::ND : MQ,
and	ND:MQ:: ID : IQ;
therefore	CD:CE :: ID : IQ;
wherefore	DH:HQ:: ID : IQ;
theref. conv. Prop. II.	AC:CB::AH : HB.

Q. E. D.

Converſely.

By Prop. IV. HD : HE :: CD : CE; theref. Sim. Eu. VI. 3, or A. CH biſects the ∠DHG;

therefore

therefore Eu. III. 7 or 8. DH = HG;
wherefore DG is bisected in F;
therefore Eu. III. 3. DG is perpendicular to AB.

Q. E. D.

PROP. VIII.

Demonstrated by Peletarius

ANALYSIS. Fig. 117, 118. *Plate* 8.

Draw DL parallel to AB meeting HE in L; join DG and let it meet AB in K.

Since EC : CD :: EH : HG,
and EC : CD :: EH : HL;
therefore HG will be equal to HL;
and therefore DK will be equal to KG;
therefore DG is perpendicular to AB;
wherefore Prop. VII. AC : CB :: AH : HB.

Q. Q. V.

SYNTHESIS.

Because AC : CB :: AH : HB,
by conv. Prop. VII. DG is perpendicular to AB;
therefore DK is equal to KG;
therefore HL is equal to HG;
wherefore EH : HL :: EH : HG:
but EC : CD :: EH : HD;
therefore EC : CD :: EH : HG.

Q. E. D.

The same by Mr. Lowry.

By conv. Prop. VII. DG is perpendicular to AB, and by Eu. III. 3. DK = KG and DH = HG; therefore CAH bisects the ∠'s DCG, DHG; wheref. Eu. VI. 3. EC:CD(CG)::EH:HG(DH).

Q. E. D.

The

The same by Mr. Swale.

ANALYSIS.

Produce GC to meet the circle in I.
It is evident that GC will be = CD;
hence, by my demon. to Prop. IV. CA : HA :: CE : HE.

Q. Q. V.

SYNTHESIS.

By analysis	CA : HA :: CE : HE:
but Eu. VI. 3.	CA : HA :: CG : GH;
therefore	CE : HE :: CG : GH,
or	CE : CG :: HE : GH:
but	GC = CD;
therefore	EC : CD :: EH : HG.

Q. E. D.

Mr. Campbell *says "the truth of this proposition is manifest from the demonstration of the last."*

PROP. IX.

Demonstrated by Peletarius.

ANALYSIS. Fig. 119, 120. *Plate* 8.

Draw EH parallel to AB meeting DF in H and the circle in K; join GK and let it meet AB in L.

By hypothesis	DC : CE :: DF : FG:
but	DC : CE :: DF : FH;
therefore	DF : FG :: DF : FH;
wherefore	FG is equal to FH;
therefore	KL is equal to LG;
therefore	the ∠GLA is a right-angle;
wherefore	the ∠GKE is a right-angle.
therefore	the ∠CDG is a right-angle;

Q. Q. V.

SYN-

SYNTHESIS.

Since	the $\angle$CDG is a right-angle,
	the $\angle$GKE will be a right-angle;
therefore	the $\angle$GLA is a right-angle;
wherefore	KL is equal to LG;
therefore	GF is equal to FK;
therefore	DF : FH :: DF : FG:
but	DC : CE :: DF : FH;
wherefore	DC : CE :: DF : FG.

Q. E. D.

The ſame by Mr. Lowry.

Draw GP perpendicular to DE meeting AB in I, and join EG; through the centre O draw SOQ parallel to DG.

Then, Eu. III. 14. PG = ED and SO = OQ; therefore Eu. I. 26. the triangles CSO, OQI being equi-angular, will be equal in all reſpects;

therefore	CS will be equal to IQ:
but	ES is equal to QG;
therefore	CE is equal to IG;
wheref. by parallels	CD : CE(IG) :: DF : FG.

Q. E. D.

The ſame by Mr. Swale.

Join GE which will evidently paſs through the centre O; through DO draw DP and join EP.

Now, the Δ's DEP, GDE having DP = GE, $\angle$DGE = $\angle$DPE and DEP = $\angle$GDE; will alſo have EP = DG; but it is evident that DF = TP; therefore FG = ET; and it is alſo evident that DF is parallel to ET;

therefore DC : CE :: DF : ET(FG).

Q. E. D.

The

The same by Mr. Campbell.

'rom E draw EH parallel to AB meeting the :le in K and DF in H; join AD, BD, EB, BG, .; draw GK meeting AB in L.

By hypothesis CDF is a right angle.
t Eu. III. 31. ADB is a right angle;
:refore $\angle$ BDG $=\angle$ EDA $=\angle$ ABE $=\angle$ BEK;
ıerefore the arc BG = arc BK and arc AG = arc AK.
ıerefore the right line BG = BK and $\angle$ ABG = $\angle$ ABK;
wherefore in the triangles BLG, BLK,
e have $\angle$ LBG = $\angle$ KBL, BG = BK and BL common;
therefore GL = LK; and therefore GF = FH;
therefore DC : CE :: DF : TH or FG.

Q. E. D.

PROP. X.

Demonstrated by Peletarius.

ANALYSIS. Fig. 121, 122, 123. *Plate* 8.

Let CG be joined meeting the circle in N; also ɔin NL, NH.

By hypothesis GM : ML :: GD : DH;
y Prop. VIII. GD : DH :: GC : CN;
ıerefore GM : ML :: GC : CN;
herefore LN is parallel to CM.
gain, by hypothesis AC : CB :: AD : DB:
:ref. conv. Prop. VII. NH will be perpen. to AB;
herefore NH is parallel to EF;
erefore the angle HNL = the angle CFE;
but the angle HNL = the angle HGL. (DGK);
erefore the angle DGK = the angle CFE.

Q. Q. F.

SYNTHESIS.

nce the angle DGK or HGL = the angle CFE.
and the angle HGL = the angle LNH;

 there-

therefore the angle LNH = the angle CFE.
By hypothesis $AC : CB :: AD : DB$;
theref. conv. Prop. VII. NH will be perpendicular to AB;
wherefore NH is parallel to EF:
but the angle LNH = the angle CFE;
therefore LN is parallel to CM;
wherefore $GM : ML :: GC : CN$;
but, Prop. VIII. $GC : CN :: GD : DH$;
therefore $GM : ML :: GD : DH$.

Q. E. D.

The same by Mr. Lowry.

Draw HC, GC meeting the circle in P, N; join NH, GP; also join NL meeting FE in R.
By Prop. VII. and Cor. NH, GP are both perpendicular to AB;
theref. (fig. 121, 123.) $\angle LRF = \angle LNH = \angle DGK = \angle CFE$;
and (fig. 122.) $\angle NRF = \angle LNH = \angle LGH = \angle DGK = \angle CFE$;
theref. in all the figures NRL is parallel to CFM;
wherefore $GM : ML :: GC : CN (CH) :: GD : DH$.

Q. E. D.

The same by Mr. Swale.

ANALYSIS.

From H, G, through E draw the diameters HQ, GT; join HL, HT, GQ, LQ, TQ, TL; and through O (the point where GQ meets AB) and M draw OMV meeting HL in V.
Now $\angle EFC = \angle DGK - \angle LGH = \angle LQH$;
but GQ is perpendicular to GH;
therefore $\angle ECF = \angle QGL = \angle QHL$.
Again $\angle LGT = \angle LHT = \angle LQT$,
and the angles QLH, QTH, QLH, GLT are right angles;
therefore $\angle QLG = \angle VML = \angle GMO$;
wherefore $GM \cdot ON = GO \cdot ML$.

Q. Q. V.

SYNTHESIS.

SYNTHESIS.

By Analysis GM : ML :: GO : ON ;
by my demon. to Prop. IX. GD : DH :: GO : ON ;
therefore GM : ML :: GD : DH ;

Q. E. D.

The same by Mr. Campbell.

Draw CG meeting the circle in N, and let DS, CW perpendicular to CA meet CG, GH in S, W; let FE, CF meet GH in I, Y, join NL, NH.

By conv. Prop. VII. NH will be perpen. to AC, and therefore ∠CWG = ∠NHG = ∠NLG : but in the triangles FYI, MYG, the angle FYI in fig. 121, 123 is common to both, and in fig. 121 = GYM, and ∠YFI = ∠YGM by *conf.*;
therefore ∠YMG = ∠YIF = ∠CWG;
wherefore ∠NLG = YMG :
therefore NL is parallel to CM;
wherefore Eu. VI. 2. GM : ML :: GC : CN
Prop. II. :: GS : SN
Eu. VI. 2. :: GD : DH ;
therefore Eu. V. 21. GM : ML :: GD : DH.

Q. E. D.

ARTICLE XXX.

Answers to the Mathematical Questions proposed in Article VI. No. I.

I. QUESTION 9, *by* Juvenis Mathematicus.

FROM the data, dear Gents, which are placed below,
The greatest of *secrets* I would have you to shew.

$$\left.\begin{array}{l} x^2+z^2-2yz+y^2=170=a \\ y^2+z^2-2zx+x^2=234=m \\ z^3+y^2-2yx+x^2=80=n \end{array}\right\}$$ Where x, y and z represent the places in the alphabet composing the secret?

Answered by Miss Sally Hill, *of* ———, *near* Birmingham.

From the sum of the given equation take the double of each, and these will result,

$$x^2-2yx+y^2-2zx+2yz+z^2=m+n-a,$$
$$x^2-2yx+y^2+2zx-2yz+z^2=a+n-m,$$
and $x^2+2yx+y^2-2zx-2yz+z^2=a+m-n$; extract the square roots, and we have

$$-x+y+z=\pm\sqrt{m+n-a},$$
$$x-y+z=\pm\sqrt{a+n-m},$$
and $x+y-z=\pm\sqrt{a+m-n}$.

Hence, $x=\frac{1}{2}(\pm\sqrt{a+m-n}\pm\sqrt{a+n-m})=11$ or 7

$y=\frac{1}{2}(\pm\sqrt{a+m-n}\pm\sqrt{m+n-a})=15$ or 3

and $z=\frac{1}{2}(\pm\sqrt{m+n-a}\pm\sqrt{a+n-m})=8$ or 4

Consequently the secret is GOD.

The same by Mr. Richard Wood, *Excise-Officer, a* Birmingham.

Take the first and second equations severally from the third, and also take the first from the second; the remainders give

$2yz-2xy=n-a$; or $(z-x)^2=((n-a)\div 2y)^2$;
$2zx-2xy=n-m$; or $(z-y)^2=((n-m)\div 2x)^2$;
$2yz-2xz=m-a$; or $(y-x)^2=((m-a(\div 2z)^2$.

Henc

Hence, by fubftitution we fhall have

$$x^2+\left(\frac{n-m}{2x}\right)^2=a\text{; or } x=\sqrt{\frac{a\pm\sqrt{a^2-(m-n)^2}}{2}}=7 \text{ or } 11;$$

$$y^2+\left(\frac{n-a}{2y}\right)^2=m\text{; or } y=\sqrt{\frac{m\pm\sqrt{m^2-(n-a)^2}}{2}}=3 \text{ or } 15;$$

$$\text{and } z^2+\left(\frac{m-a}{2z}\right)^2=n\text{; or } z=\sqrt{\frac{n\pm\sqrt{n^2-(m-a)^2}}{2}}=4 \text{ or } 8.$$

Otherwife by Mr. Ralph Simpfon.

Given $x^2+(z-y)^2=170=49+121$,
$y^2+(z-x)^2=234=225+9$,
and $z^2+(y-x)^2=80=16+64$.

Hence, it is evident that $x=7$ or 11, $y=15$ or 3, and $z=4$ or 8, the fame as above.

Ingenious Solutions were alfo given by Meffrs. Burdon, Elliot, Harris, Mabbot, Orpheus *and* Swale.

II. QUESTION 10, *by Mr.* T. Bulmer, Sunderland.

There is a cone, which being fufpended by its vertex, the number of vibrations it makes in a minute, its altitude, and the radius of its bafe in inches, are as 11, 10 and 1 :—Required how often it vibrates in a minute and its folid content ?

Anfwered by Orpheus, *of* Hamfterly.

Put $x=$ the altitude, $y=$ the radius of the bafe, $a=60''=$ 1 minute, and $l=39{\cdot}13$ inches, the length of a pendulum vibrating feconds.

Then by the principles of fluxions, the diftance of the centre of ofcillation from the point of fufpenfion will be $(4x^2+y^2)\div 5x$; hence the nature of pendulums

$\sqrt{(4x^2+y^2) \div 5x} : \sqrt{l} :: a : a\sqrt{5lx \div (4x^2+y^2)}$, the number of vibrations made by the cone in one minute. Whence by the queſtion

$a\sqrt{5lx \div (4x^2+y^2)} : x :: 11 : 10$, and $x : y :: 10 : 1$; taking means and extremes we have

$10a\sqrt{5lx \div (4x^2+y^2)} = 11x$, and $10y = x$; theſe equations, by proper reduction, give

$y = \sqrt[3]{50al \div 48521} = 5{\cdot}25554$ the radius of the cone's baſe, and $x = 52{\cdot}5554$ its altitude. Hence the ſolidity of the cone is $1520{\cdot}14967$ inches, and it makes $57{\cdot}8109$ vibrations in one minute.

W. W. R.

The ſame by Mr. Burdon, Acaſter-Malbis.

If x repreſent the radius of the baſe of the cone, $10x$ and $11x$ will repreſent its altitude and the number of vibrations it makes in one minute, and $401x \div 50$ will be the diſtance of the centre of oſcillation from its vertex; alſo $60 \div 11x$ is = the time in ſeconds of one vibration. Hence, by the known laws of pendulums it will be $\sqrt{39{\cdot}2} : 1 :: \sqrt{401x \div 50} : 60 \div 11x$, from which analogy x is found $= 5{\cdot}258$ inches; conſequently the other requiſites are eaſily had.

The ſame anſwered by Mr. Richard Wood.

Put $39{\cdot}2 = b$, $60'' = a$, $11x =$ the number of vibrations the cone makes in one minute, and $x =$ the radius of its baſe; then *(page 438, Hodgſon's Fluxions)* $8x$ will be the diſtance of the centre of oſcillation from the vertex; and the number of vibrations made by the cone in one minute will be expreſſed by $\sqrt{ba^2 \div 8x} = 11x$ by the queſtion. Hence

Hence $x = \sqrt[3]{ba^2 \div 968} = \cdot 265$ inches, the radius of the base of the cone; from which its altitude, number of vibrations it makes in one minute, and its solid content are readily found.

The same by Mr. James Boyce, Birmingham.

Let x, $10x$, and $11x$ denote the radius of the base of the cone, its altitude, and the number of vibrations it makes in a minute respectively. Then by pendulums $(11x)^2 : (60)^2 :: 39\cdot 2 : 39\cdot 2 \times 3600 \div 121x^2$ the length of the pendulum, or the distance from the vertex of the cone to its centre of oscillation: but by the principles of fluxions this distance is also found to be $401x \div 50$; therefore making these two expressions equal to one another, we shall find x, &c. as above.

True Solutions were given by Messrs. Bulmer, Dawes, Harris, Lowry, Simpson *and* Swale.

III. QUESTION 11, *by Mr.* J. Surtees, Sunderland.

It is required to find the pressure of water with the velocity of $0\cdot 00002$ feet per second, against a flood-gate placed perpendicular to the horizon, whose breadth is 18, and depth 12 feet?

Answered by Mr. John Surtees.

Let $2n = 12$ feet, $3n = 18$, $v = 0\cdot 00002$, $s = 16\frac{1}{2}$ = the space descended by a falling body in one second, and $a = 62\frac{1}{2}$ lbs. = the weight of a cubic foot of water. Then (*prop.* 107, *Emerson's Mechanics*) $6an^2 \times v \div 2s$ = the force of the stream: but (*cor.* 2nd. *ibid*) if any part of the fluid lie upon (or be sustained by) the plane, the force will be augmented by the weight (or pressure) of so much water. And (*by prop.* 91, *cor.* 2.) the quantity of pressure

on

on any plane surface sustaining a fluid is equal to that of the same plane placed parallel to the horizon, at the depth where its centre of gravity is: therefore $6an^3 =$ the quantity of its pressure, and $6an^3 + 6an^2 \times v \div 2s = 81000 \cdot 000000016785$ lbs. the whole pressure against the gate; and that pressure is also equal to the resistance the same plane would meet with in moving with the given velocity and driving the fluid before it.

The same answered by Mr. Burdon.

Mr. Emerson, at page 133 of his Hydrostatics, 8vo. says, "the force of a stream of water against any plane obstacle at rest, is equal to the weight of a column of water, whose base is the section of the stream, and height the space descended through by a falling body to acquire that velocity." Now the spaces described by falling bodies being as the squares of the velocities, we have

$\left(32\frac{1}{6}\right)^2 16\frac{1}{12} :: \left(\frac{1}{50000}\right)^2 : \frac{3}{482500000000}$ feet, the height fallen through to acquire the velocity of the water; therefore $18 \times 12 + \frac{3}{482500000000} \times 62\frac{1}{2} =$ $\frac{81}{965000000}$ lbs. the force against the gate; this added to 81000 the pressure (*vid. Qu.* 2. *Yorks. Rep.*) gives $81000\frac{81}{965000000}$ lbs. the pressure in this case.

Answers to this Question were also received from Orpheus *and* J. H. Swale.

IV. QUESTION 12, *by Mr. J.* Rutherford, Weardale.

On Midsummer-day, in latitude 54° 40′ north, at 10 o'clock in the forenoon, I observed the sun to shine into a shaft made for the purpose of winding up

up the ore got in the mines; the declination of the ſhaft I found was S. S. W. and breadth 4 feet:— *Query* the depth to which the ſun ſhined therein, and the length of the higheſt ray from the upper edge of the ſhaft, to the loweſt point enlightened thereby on the oppoſite ſide?

Anſwered by Mr. Surtees.

By ſpherics the apparent altitude of the ſun's upper limb is found to be 51° 54′ very near, and his azimuth from the ſouth = 47° 38′ 43″. Then in (fig. 124, pl. 8.) where AD repreſents the ſhaft, CB a line croſſing the ſhaft in direction of a ray proceeding from the ſun, ∠CAB a right angle, there is given AC = 4 feet, and ∠CBA = 47° 38′ 43″ + 22° 30′ = 70° 8′ 43″, to find CB = 4·253 feet: then, as coſine alt. (51° 54′) : BC :: ſine alt. 5·424 feet = the depth to where the ſun will ſhine; and as coſine alt. : BC :: rad. : 6·892 feet = the length of the longeſt ray.

The ſame by Mr. John Dawes, Birmingham.

By ſpherics the ſun's azimuth from the ſouth = 47° 40′, and allowing for refraction his altitude is found = 51° 39′ 40″.

Let ⊙ABC (fig. 125, pl. 9.) repreſent the ſhaft, a⊙b its declination = 22° 30′, ⊙a, a meridional line, a⊙e the ſame azimuth of the ſouth, and ⊙d the ray required. Then in the rt. ∠'d, △⊙Ac, there are given ⊙A = 4 and ∠A⊙c (= e⊙g = the comp. of (b⊙a + a⊙e) to find ⊙c. Again in the rt. ∠'d⊙cd, there are given ⊙c (found above) and the ∠c⊙d (the ſun's altitude) to find ⊙d = 6.855 feet, the length of the longeſt ray, and cd = 5·378 feet, the depth to which the ſun ſhone into the ſhaft.

The

The same by Mr. William Burdon.

Conceive a spherical triangle formed by Z the zenith, S the sun, and P the pole, in which are given ZP=35° 20′ the colat. PS=66° 32′ the co-declination, and the included angle ZPS=30° or 2 hrs. the time from noon, to find ∠Z=132° 19′ 47″ the sun's azimuth from the north, and ZS =38° 20′ 42″ the colat: hence 51° 39′ 18″ the true alt. of the sun's centre, or by allowing for semidiameter, refraction, and parallax, gives 51° 55′43″ the apparent altitude of the sun's upper limb. Then in fig. 126, pl. 9, AD=4 feet, the breadth of the shaft, ∠ABD=S. S. W.+supp. 132° 19′ 47″= 70″ 10′ 13″ and the angle, DBC=51° 55′ 43″ the sun's altitude. In the rt. ∠'d△ADB, is found DB =4·2521; also in the rt. ∠'d△DBC is found CD =5·4285 feet, the depth the sun shone within the shaft, and BC=6·8955 feet, the length of the longest ray.

The same by Mr. Rutherford, *the Proposer.*

From the *data*, I find the sun's azimuth from the south=47° 40′ 13″12‴ and his altitude=51° 39′18″ 26‴. Then in (fig. 127, pl. 9.) let S, W, N, E, represent South, West, North, East, and Cn the breadth of the shaft=4 feet, and the angle AC n=19° 49′ 46″ 48⁗ Hence by plane trigonometry

s.∠ACn : : Cn : : rad. : CA=4·261931 feet.

s.∠CBA : CA : : rad. :BC=6·869 feet, the longest ray, and rad. : BC:: s. ∠BCA : BA=5·387 feet, the depth that the sun shined into the shaft.

W. W. R.

Other ingenious Solutions were given by Messrs. Boyce, Harris, Orpheus, Simpson, Swale *and* Wood.

V. QUES-

V. QUESTION 13. *by Mr.* Burdon, Acaster-Malbis.

There are three towns A, B and C, the roads to which, from one another, form a right-angled triangle. Now a person had to travel from the town B at the right angle to A; but, after going two miles, had occasion to call somewhere on the road from A to C; he therefore takes the nearest way to it, and then finds he is one mile and a half from A and three from C:—*Query* the distance from B to C, and the number of miles he travelled when arrived at A?

Answered by Mr. Rd. Elliot.

Conf. Upon the given distance of the towns A and C (fig. 128, plate 9.) describe a semi-circle, and at D, the place where the traveller had to call, erect the perpendicular DE equal to the side of a square, whose area is equal to the rectangle AD.AC; make DF equal to half the distance he travelled from B before he changed his direction, and join EF; with the centre A and distance EF—DF describe an arc to cut ED in G; and produce AG to meet the semi-circle in B; then B will represent the town from which he began his journey.

Demon. We have only to prove that $BG=2DF$

in order to which we have	$ED^2+DF^2=EF^2$,
and by *conf.*	$(AG+DF)^2=FE^2$;
hence,	$AG\cdot(AG+2DF)=DE^2$:
but by *conf.*	$AD\cdot AC=DE^2$;
therefore	$AG\cdot(AG+2DF)=AD\cdot AC$;
or	$AG:AD::AC:AG+2DF$:
but by sim. $\triangle$'s,	$AG:AD::AC:AB$;
wherefore	$AB=AG+2DF$;
consequently	$BG=2DF$.

Q. E. D.

From above, by calculation, BC is found $=2\cdot4358$, and $BG+DG+AD=4\cdot4656$, the number of miles travelled.

The

The same by Orpheus, *of* Hamsterly.

Let ABC (fig. 128, plate 9.) be the right-angled triangle formed by the roads, G the place where the traveller left the road AB, and D the place he had to call at on the road AC; it is evident from the nature of the question, that GD must be perpendicular to AC, and therefere the right-angled triangle ADG is similar to the right-angled triangle ABC. Now put $BG = 2 = a$, $AD = 1\frac{1}{2} = b$, $AC = AD + DC = 4\frac{1}{2} = c$ and $AG = x$.

Then $AG : AD :: AC : AB = bc \div x$: but $AB = BG + GA = a + x = bc \div x$; hence $x^2 + ax = bc$; solved gives $x = 1{\cdot}7838$; whence BC is easily found $= 2{\cdot}4357$ miles, and $BG + GD + DA = 4{\cdot}46537$ the number of miles required.

Solutions equally ingenious with those given above, were received from Messrs. Boyce, Burdon, Dawes, Harris, Mabbot, Simpson, Swale *and* Wood.

VI. QUESTION 14, *by* Plus-Minus Selby.

A person of my acquaintance has an equilateral triangular yard to be divided into three parts by paling, drawn from the center of a bason, somewhere within it, to the nearest point in each side. Now he is informed that it will cost him as much doing as 12s. 6d. per pole, as the whole yard would paving at 9d. per yard:—*Query* the sides and area of the said yard?

Answered by Mr. Richard Elliot.

By a well-known property of the equilateral triangle, the quantity of paling is equal to the perpendicular of the triangular yard. And the perpendicular

cular of any equilateral triangle, is to the area, as unity, to half the base; which by the question will be in the inverse ratio of the prices; that is, as 1 : $\frac{1}{2}$ base :: 9 : 150 ÷ 55; hence the base (or side) itself is equal to 6·0606 yards, and the area of the yard is equal to 15·9045 square yards.

The same by Mr. John Harris, *Teacher of the Mathematics, at* Caermarthen.

Put x = the side of the equilateral triangle; then (Eu. I. 47) $\frac{1}{2}x\sqrt{3}$ = the perpendicular = the sum of the perpendiculars drawn from any point within the triangle to the sides, and $\frac{1}{4}x^2\sqrt{3}$ = the area of the triangle; hence $\frac{1}{2}x\sqrt{3} \times 12·5 \div 5·5 = 3x^2\sqrt{3} \div 16$; which reduced gives $x = 6·0606$, &c. the side of the triangle, and $\frac{1}{2}x\sqrt{3} = 5·2486$, &c. the perpendicular; and therefore the area is = 15·9049, &c. square yards.

Proof.

5·2486 &c. yards at 12s. 6d. per pole is 11s. 11·144d. and 15·9049, &c. yards at 9d. per yard is 11s. 11·144d.

Ingenious answers to this question were also received from Messrs. Boyce, Burdon, Dawes, Elliot, Orpheus, Simpson, Swale *and* Wood.

VII. QUESTION 15, *by Mr. Collin Campbell, Kendal.*

If PGHI, KCMI. (fig. 31.) be two wheels, revolving round the centres S, O, and connected by the flexible band FGHMLKF. It is required to determine the friction of that band on each wheel, supposing the centre S fixed, and the centre O urged by force in the direction SO ÷ T.

Answered by Mr. Burdon.

The whole pressure upon the surface of the wheel KCML (fig. 31, pl. 1.) will be easily found to be =

T$\times$arc KLM$\div$2KO and that on the wheel FGHI $=$T$\times$arc FGH$\div$2SH, which are as the effects of the friction on each wheel; or since the arc FGH is similar to the arc KCM, the friction on the wheel FGHI is to that on the wheel KCML, as the arc FGH to its supplement FIH.

VIII. QUESTION 16, *by Mr. J. Fletcher, Liverpool.*

Seeing an exciseman's staff in form of a cylinder, three-fourths of an inch in diameter, and thirty-six inches long, immersed in a vessel of beer at one end, the other resting on the edge of the vessel 3 inches above the liquor, I observed 13 inches along the staff's axis to be dry:—Required the weight of the staff? a cubic inch of beer weighing 0·5949 oz. aver.

Answered in Gent. Diary, 1790.

IX. QUESTION 17, *from Lawson on the Ancient Analysis.*

Let there be a triangle ABC, whose base BC is bisected in D, and through the vertex A a line AE drawn parallel to BC, and any line drawn through D to meet AB, AC, AE in F, G, H; then I say GD : DF :: GH : HF :—Required the demonstration?

Answered in Article XXIX.

X. QUESTION 18, *from the same.*

If in AB the diameter of a circle two points C and D be taken such that AC : CB : : AD : DB, and through the point D any line be drawn to meet the circle in E and F, and CE, CF be joined; then I say EC : CF : : ED : DF :—Required the demonstration?

Answered in Article XXIX.

XI.

XI. QUESTION 19, *from Stewart's General Theorems.*

Let there be any number of given points A, B, C, &c. and let a, b, c, &c. be given magnitudes as many in number as there are given points; a point X may be found, ſuch, that if from the given points A, B, C, &c. there be drawn right lines to the point X, and from the given points and the point X there be drawn right lines to any point Y, the ſquare of AY together with the ſpace to which the ſquare of BY has the ſame ratio that a has to b, together with the ſpace to which the ſquare of CY has the ſame ratio that a has to c, and ſo on, will be equal to the ſquare of AX together with the ſpace to which the ſquare of BX has the ſame ratio that a has to b, together with the ſpace to which the ſquare of CX has the ſame ratio that a has to c, and ſo on, together with the ſpace to which the ſquare of XY has the ſame ratio that a has to the ſum of a, b, c, &c.—Required the demonſtration?

Anſwered in Article XXXIV.

XII. QUESTION 20, *by Mr. R. Simpſon.*

Bartered a piece of broad cloth, containing a yards at b ſhillings per yard, for a piece of fine Iriſh linen and another of cambric. Now the ratio of the yards in theſe two pieces was that of c to d, and the ratio of their values per yard, in ſhillings, that of m to n; alſo the rated price of the linen per yard, was to the number of yards in the piece as r to s:—Required the yards, prices per yard, and values of the two pieces?

Anſwered by Mr. Harris.

Let cx = the number of yards of Iriſh linen
dx = the number of yards of cambric
my = the value of one yard of the linen, and

 ny =

ny=the value of one yard of cambric. Then
$cmyx$=the whole value of the linen, and
$dnyx$=the whole value of the cambric, hence, by the queſtion $cmyx+dnyx=ab$,
and $my:cx::r:s$; or $crx=msy$;
theſe equations being reduced, gives

$x=\sqrt{absm\div rc\cdot(cm+dn)}$, and

$y=\sqrt{abrc\div ms\cdot(cm+dn)}$. Hence

$\sqrt{absmc\div r\cdot(cm+dn)}$=the number of yards of iriſh,
$d\sqrt{absm\div rc\cdot(cm+dn)}$ = the number of yards of cambric
$\sqrt{abcrm\div s\cdot(cm+dn}$ = the value per yard of the liner
$n\sqrt{abrc\div ms\cdot(cm+dn)}$ = the value per yard of the cambric
$abcm\div(cm+dn)$ = the value of the iriſh linen, and
$abdn\div(cm+dn)$ =that of the cambric.

The anſwers by Meſſrs. Burdon, Elliot, Orpheus, Mabbott, Simpſon *and* Swale, *were nearly the ſame as that above.*

XIII. QUESTION 21, *by Mr.* Olinthus Gilbert Gregory.

The axis of a ſphere is 12 inches; what is the difference between the ſolidity of this ſphere, and that of a cone whoſe ſlant height is to the radius of its baſe as 3 to 1, and the whole ſurface equal to the ſurface of the ſphere?

Anſwered by Mr. James Boyce.

The whole ſurface of a cone whoſe ſlant height is 3, and the radius of its baſe 1 inch, is $=3.1416\times4$, and the ſurface of the ſphere is $=3\cdot1416\times144$. Hence, the ſurfaces of ſimilar ſolids being as the ſquares of their like dimenſions, it will be as $\sqrt{3\cdot1416\times4}:\sqrt{3\cdot1416\times144}::1:6$ inches, the radius of the cone's baſe, and its ſlant height is $=18$ inches. Whence

the solidity of the sphere is $=904\cdot8638$ inches,
the solidity of the cone is $=639\cdot7538$ inches;
ergo, the required difference is $=265\cdot11$ inches.

Algebraical solutions to this question were received from Messrs. Burdon, Dawes, Elliot, Gregory, Harris, Orpheus, Simpson, Swale *and* Wood.

XIV. QUESTION 22, *by Mr.* John Lowry.

If tangents be drawn from the extremities of a given oblique parabola: it is required to determine the area of the greatest ellipsis that can be inscribed in the space included between the tangents and the curve?

Answered by Mr. John Lowry.

Const. Let POQ (fig. 129, pl. 9.) be the given parabola, and PG, QG, the tangents drawn from its extremities P, Q intersecting each other in G; draw the diameter GOH, also parallel to QG, PG let OC, OD be drawn; then (*by Prop.* 70, *Book* 3, *of Emerson's Conics*) describe an ellipsis OCbD to pass through the points O, C, D and touch the tangents in C, D; then I say OCbD will be the ellipsis required.

Demon. Through O draw the tangent EOF, which by the property of the parabola, will be bisected in O, and parallel to PHQ; then, by the scholium to Theorem VIII, *Simpson* on the *Maxima et Minima* of Geometrical quantities, the triangle EGF will be a *maximum*. Again, by reason of parallels, EC=CG and FD=GD; whence it is manifest from the *scholium* just mentioned, that the ellipsis OCbD is the greatest that can be inscribed in the triangle EGF, and therefore the greatest that can be inscribed in the space PGQOP.

Calculation. FC, ED, being joined, will, by the property of the ellipsis, interfect each other in R, its centre; alfo CD being joined (meeting the diameter in S) will be equal and parallel to EO; and SG will likewife be equal to SO;

and, by the property of the $\triangle$, OR or $Rb = \frac{1}{2}GR$;

therefore $Ob = GR = \frac{2}{3}OG$:

but, by Em. Conics II. 20, $OG = OH$;

therefore $Ob = \frac{2}{3}OH$,

Again, let the conjugate IRV be drawn parallel to EOF meeting the tangent PG in L;

then $RL = \frac{2}{3}EO$ and $CS = \frac{1}{2}EO$;

theref. Em. Con. I. 47, cor. 1. $IR^2 = CS \cdot RL = \frac{1}{3}EO^2$;

hence, $IR = EO \div \surd 3$ and $IV = 2EO \div \surd 3 = PH \div \surd 3$;

therefore the area of the ellipsis, is to the area of the parabola, as $3 \cdot 1416$ is to $8\surd 3$; or to the area of the triangle EGF as $3 \cdot 1416$ to $3\surd 3$.

Q. E. D. et I.

An ingenious folution to this queftion was alfo given by Mr. Richard Elliot, *of* Liverpool.

XV. QUESTION 23, *by Mr.* Lowry.

Given the perimeter, the vertical angle and area of a fpherical triangle to determine it?

Anfwered by Mr. Lowry.

Analyfis. Fig. 130, *Plate* 9.

Suppofe the thing done and that ABC is really the triangle required; produce the fides CA, CB to E, D, fo that CE, CD may be each equal to half the given perimeter; perpendicular to CE, CD draw the arches EO, DO to interfect in O; and about O as a pole, with the diftance OE or OD defcribe the lefs circle EPD, which, as is well known, will touch the bafe AB in fome point as P; draw the arches AO, BO.

Becaufe

Becauſe the area and vertical angle are given, the ſum of the angles at the baſe will be given; therefore the ſum of their ſupplements will be given, *i. e.* the ſum of the angles EAB, DBA will be given.

Again, the right angled triangles AOE, APO, having EO=OP, ∠OEA=∠APO=a right angle, and AO common to both, will be equal and ſimilar in every reſpect;

therefore ∠OAP=½∠EAB;
in like manner ∠OBP=½∠DBA;
therefore the ſum of the angles OAP, OBP is given.

Hence in the triangle ABO there will be given the ∠AOB=½∠EOD, the perpendicular OP, and and the ſum of the angles OAB, OBA; or in the ſupplemental triangle there will be given the baſe, the perpendicular, and the ſum of the ſides to determine the triangle; and this is done at prop. VIII. Art. III. No. I.

If the baſe, ſum of the ſides, and area had been given, then in the ſupplemental triangle there would have been given the vertical angle, the ſum of the angles at the baſe, and the perimeter, which is the ſame as the above.

Mr. J. H. Swale, *of* Leeds, *anſwered this queſtion.*

XVI. QUESTION 24, *by Mr.* W. Pearſon, *North Shields.*

The fluent of $\overline{a+cz^n}|^m \times z^{pn-1}\dot{z}$ being given, from p. 94. of *Simpſon's Fluxions*, it is required to find the fluents of $\overline{a+cz^n}|^{m-r} \times z^{pn+vn-1}\dot{z}$, of $\overline{a+cz^n}|^{m+r} \times z^{pn-vn-1}\dot{z}$, and alſo of $\overline{a+cz^n}|^{m-r} \times z^{pn-vn-1}\dot{z}$?

Anſwered

$$-\frac{\overline{s+1}\cdot QR}{\overline{q+1}\;a}-\frac{s+2}{q+2}\cdot\frac{Scz^n}{a}-\frac{s+3}{q+3}\cdot\frac{Tcz^n}{a}(v)$$

$$+\frac{{}^{*}\overline{t+1}\cdot\overline{t+2}\cdot\overline{t-3}\cdot\overline{t-4}\cdot\overline{t-5}\,(r+v)}{m\cdot\overline{m-1}\cdot\overline{m-2}\,(r)+\overline{p-1}\cdot\overline{p-2}\cdot\overline{p-3}\,(v)}\times\frac{\overline{-C}|^{v}A}{a^{r+v}};$$

which is the ſame as determined by Mr. Simpſon on page 395 of his Fluxions. Where H, I, K, L, ...R, S, T, &c. denote the terms immediately preceding thoſe where they ſtand under their proper ſigns, R being the laſt term of the firſt ſeries; Q &c. as mentioned above.

*It may perhaps be neceſſary to explain, how this expreſſion, was obtained.

Since the numerator $\overline{t-1}\cdot\overline{t-2}\cdot\overline{t-3}\cdot\overline{t-4}\,(v)$ $\times\overline{p'+m}\cdot\overline{p'+m-1}\cdot\overline{p'+m-2}(r)$ of the laſt term of the fluent (by ſubſtituting for p' &c.) is $=\overline{t-1}\cdot\overline{t-2}\cdot\overline{t-3}$ $(v)\times\overline{p-v+m}\cdot\overline{p-v+m-1}\cdot\overline{p-v+m-2}\,(r)$; where the firſt factor of the ſecond progreſſion $(p-v+m)$ is leſs by unity than the laſt factor $(t-v)$ of the firſt progreſſion; it is evident that the ſaid ſecond progreſſion is only a continuation of the firſt to r more factors; and ſo the laſt term of the fluent where A is found, is truly expreſſed by $+\frac{\overline{t-1}\cdot\overline{t-2}\cdot\overline{t-3}\,(v+r)}{m\cdot\overline{m-1}(r)\times\overline{p-1}\cdot\overline{p-2}(v)}$ $\times\frac{\overline{-C}|^{v}A}{a^{r+v}}$.

By a ſimilar method of reaſoning, (uſing Prob. 6. inſtead of 7) the value of F (the fluent of $(a+cz^n)^{m-r}z^{pn-vn-1}\dot{z}$ will alſo be $=$

$$Q\frac{{}^{m-r+1}z^{pn-vn}}{\overline{q+1}\cdot na}-\frac{s+2}{q+2}\cdot\frac{Hcz^r}{a}-\frac{s+3}{q+3}\cdot\frac{Icz^n}{a}(v)$$

+

$$\times\left(\frac{z^{-vn}}{\overline{q+1}\cdot na}-\frac{\overline{s+2}\cdot cz^{n-vn}}{\overline{q+1}\cdot\overline{q+2}\cdot na^2}+\frac{\overline{s+2}\cdot\overline{s+3}\cdot c^2 z^{2n-vn}}{\overline{q+1}\cdot\overline{q+2}\cdot\overline{q+3}\cdot na^3}(v)\right)$$

$$\pm^{*}\frac{p'+m}{m}\cdot\frac{p'+m-1}{m-1}(r)\times\frac{1}{a^r}\times\frac{t-1}{p-1}\cdot\frac{t-2}{p-2}\cdot\frac{t-3}{p-3}(v)\times\frac{c^v A}{a^v}.$$

Where $q=p-v-1$; $s=m+q=m+p-v-1$, $t=p+m+1$.

* $+$ or $-$, according as v is even or odd.

If the last term of the first series exclusive of the multiplicator $Q^m z^{pn}$ be denoted by β, the multiplicator $\frac{p'+m}{m}\cdot\frac{p'+m-1}{m-1}\cdot\frac{p'+m-2}{2n-2}(r)\times\frac{1}{a^r}$ to the second series will be $=(p'+m).n\beta$ (Art. 290.); and therefore the first term of this series, including its multiplicators is $=\frac{\overline{p+m}\cdot n\cdot\beta Q^{m+1} z^{pn-vn}}{\overline{q+1}\cdot na}$

$$=\frac{p'+m.\ \beta\, Q^{m+1} z^{pn-vn}}{\overline{q+1}\cdot a}$$

$$=\frac{\overline{s+1}\cdot\beta\, Q^{m+1} z^{pn-vn}}{\overline{q+1}\cdot a}$$

which, if R be put to denote the last term of the first series $\left(\beta Q^m z^{pn-vn}\right)$, will be expounded by $-\frac{s+1}{q+1}\cdot\frac{QR}{a}$.

Hence it follows that the fluent of $(a+cz^n)^{m-r} z^{pn-vn-1}\dot{z}$ given above, will also be equal to

$$-\frac{Q^{m-r+1} z^{pn-vn}}{\overline{f+1}\cdot na}+\frac{g+1}{f+2}\cdot\frac{QH}{a}+\frac{g+2}{f+3}\cdot\frac{QI}{a}(r)$$

—

If room would permit, a great many curious forms of fluxions with their correſponding fluents might be exhibited. It will alſo now appear pretty plain how the other required fluents may be found; Mr. *Simpſon* has given us the fluents in one form for each, and I ſhall now put the ſame down in another form.

The fluent of $(a+cz^n)^{m-r} z^{pn+vn-1}\dot{z}$ is

$$\frac{Q^{m-r+1} z^{pn+vn}}{\overline{s+1}\, n\dot{c}z^n} - \frac{q}{s}\cdot\frac{aH}{cz^n} - \frac{q-1}{s-1}\cdot\frac{aI}{cz^n}(v)$$

$$-\frac{pR}{f+1} + \frac{g+1}{f+2}\cdot\frac{QS}{a} + \frac{g+2}{f+3}\cdot\frac{QT}{a} + \frac{g+3}{f+4}\cdot\frac{QV}{a}(r)$$

$$+\frac{p\cdot\overline{p+1}\cdot\overline{p+2}}{t\cdot\overline{t+1}\cdot\overline{t+2}}(v)\times\frac{\overline{p+m}\cdot\overline{p+m-1}\cdot\overline{p+m-2}}{m\cdot\overline{m-1}\cdot\overline{m-2}}(r)\times\frac{a^{v-r}A}{\overline{\overline{-c}}\rceil^{v}}$$

Where H, I, K, L,...R, S, T, &c. denote the terms immediately preceding thoſe where they ſtand under their proper ſigns; R being the laſt term of the firſt ſeries, alſo $Q=a+cz^n$, $q=p+v-1$, $s=q+m-r$ $=p+v-1+m-r$, $t=p+m-r+1$, $f=m-r$ and $g=p+m-r$.

The fluent of $(a+cz^n)^{m+r} z^{pn-vn-1} z$ is

$$\frac{Q^{m+r} z^{pn-vn}}{gn} + \frac{f}{g-1}\cdot\frac{aH}{Q} + \frac{f-1}{g-2}\cdot\frac{aI}{Q}(r)$$

$$+\frac{\overline{m-1}\cdot R}{q+1} - \frac{s+2}{q+2}\cdot\frac{cz^nS}{a} - \frac{s+3}{q+3}\cdot\frac{cz^nT}{a}(v)$$

$$+\frac{m+1}{p'+m+1}\cdot\frac{m+2}{p'+m+2}(r)\times\frac{t-1}{p-1}\cdot\frac{t-2}{p-2}(v)\times\frac{\overline{\overline{-c}}\rceil^{v}A}{a^{v-r}}$$

Where

Where $p' = p - v$, $f = m + r$, $g = p' + m - r$, $q = p - v - 1$, $s = m + q$, $t = p + m + 1$, and the reſt of the letters as before.

The fluent of $(a + cz^n)^{m+r} z^{pn+vn-1} \dot{z}$ is

$$\frac{Q^{m+r+1} z^{pn+vn}}{\overline{s+1} \cdot ncz^n} - \frac{q}{s} \cdot \frac{aH}{cz^n} - \frac{q-1}{s-1} \cdot \frac{aI}{cz^n} \, (v)$$

$$+ \frac{paR}{gQ} + \frac{f}{g-1} \cdot \frac{aS}{Q} + \frac{f-1}{g-2} \cdot \frac{aT}{Q} + \frac{f-2}{g-3} \cdot \frac{aV}{Q} \, (r)$$

$$+ \frac{p \cdot \overline{p+1} \cdot \overline{p+2} \, (v) \times \overline{m+1} \cdot \overline{m+2} \, (r)}{\overline{p+m+1} \cdot \overline{p+m+2} \cdot \overline{p+m+3} \, (v+r)} \times \frac{a^{v+r} A}{\overline{-c}|^{v}};$$

where $Q = a + cz^n$, $q = p + v - 1$, $s = q + m + r$, $t = p + m + r + 1$, $f = m + r$, $g = p + m + r$, HI, &c. as before, which is a different *form* from that put down by Mr. *Simpſon* on page 322 of his Fluxions.

XVII. QUESTION 25, *anſwered by* Pappus Junior.

Upon AB (fig. 131, pl. 9.) deſcribe a ſemicircle; and let CF, perpendicular to AB, meet the ſemicircle in F; join AF, and draw DG parallel to CF, meeting the ſemicircle in G, and AF in H; and let the rectangle KDH be equal to the ſquare of CF.

Becauſe	$KD \cdot DH = CF^2$,
it will be	$HD : CF :: CF : DK$;
but	$HD : CF :: AD : AC$;
therefore	$AD : AC :: CF : DK$;
wherefore	$AD^2 : AC^2 :: CF^2 : DK^2$;
but	$AD^3 : AD \cdot AC^2 :: AD^2 : AC^2$;
therefore	$AD^3 . AD \cdot AC^2 :: CF^2 : DK^2$:

X but

If room would permit, a gr… $: AD \cdot CB : AC \cdot CB$,
of fluxions with their cor… $CB = CF^2$;
be exhibited. It will al… $: AC^3 :: AD \cdot CB : CF^2$;
how the other requir… $AD^3 : AC^3 :: AD \cdot CB : DK^2$;
Simpson has given … CF^2 is greater than $GD \cdot DK$,
each, and I shal… will be greater than $GD \cdot DH$,
form. KD is greater than GD,
KD^2 is greater than GD^2;
The flu… $DG^2 = AD \cdot DB$;
DK^2 is greater than $AD \cdot DB$;
Q… the ratio of $AD \cdot CB : DK^2$ is less than the ratio of $AD . CB : AD \cdot DB$, that is, less than the ratio of $CB : BD$,
$AD^3 : AC^3 :: AD \cdot CB : DK^2$;
therefore the ratio of $AD^3 : AC^3$ is less than the ratio of $CB : BD$.

Q. E. D.

The same otherwise by Mr. Lowry.

On AB (fig. 131, pl. 9.) as a diameter describe the circle AFBP, and draw CFP perpendicular to AB: join AF, AP.

By hypothesis $AD^3 \div AC^3$ is less than $CB \div BD$;
therefore $AC^3 \cdot CB$ is greater than $AD^3 \cdot DB$;
that is, $AC^3 \cdot CB$ must be a maximum.

Now by the circle $AC : CF :: CF : CB$;
therefore $AC^3 : AC^2 \cdot CF : CF : CB$;
therefore $AC^3 \cdot CB = AC^2 \cdot CF^2$;
wherefore $AC^2 \cdot CF^2$, or $AC \cdot CF$, or the triangle APF must be a maximum.

But the greatest triangle that can be inscribed in a circle, is well known to be the equilateral one; therefore $AC = \frac{3}{4} AB = 3BC$; consequently the truth of the proposition is manifest.

XVIII.

QUESTION 26, *answered by Mr.* Lowry.

$$\dot{x}+ry^{-1}\dot{y}=y^{-n}x^{m}\dot{x}\div a;$$

... $r^{-1}x^{(np\div r)}y^{n}$, and we get

$$x^{(np\div r)-1}\dot{x}y^{n}+nx^{(np\div r)}y^{n-1}\dot{y}=nx^{m+(np\div r)}\dot{x}\div ra;$$

the flu. give $x^{(np\div r)}y^{n}=nx^{m+(np\div r)+1}\div ra\cdot(m+(np\div r)+1)$

Cor. 1. When $n=r$; $x^{p}y^{r}=x^{m+p+1}\div a\cdot(m+p+1)$; the same as at art. 262, *Simpson's* Fluxions.

Cor. 2. If $p=r=1$; $x^{n}y^{n}=nx^{m+n+1}\div a\cdot(m+n+1)$; the same as in question 963, *Ladies' Diary*.

Cor. 3. By a similar method, the relation of x and y may be determined, in the equation,

$$ry^{-1}\dot{y}-px^{-1}\dot{x}=y^{-n}x^{m}\dot{x}\div a;$$

for multiply by $nx^{(np\div r)}y^{n}\div rx^{(2np\div r)}$, it becomes

$$nx^{(np\div r)}y^{n-1}\dot{y}-npr^{-1}x^{(np\div r)-1}\dot{x}y^{n})\div x^{(2np\div r)}=nx^{m-(np\div r)}$$
$$\dot{x}\div ra;$$

the flu. give $y^{n}x^{(np\div r)}=nx^{m-(np\div r)+1}\div ra\cdot(m-(np\div r)+1$.

Cor. 4. When $r=n$; $y^{r}\div x^{p}=x^{m-p+1}\div a\cdot(m-p+1$.

Cor. 5. If $p=r=1$; $y^{n}\div x^{n}=nx^{m-n+1}\div a\cdot(m-n+1)$.

The same by Mr. John Dawes, Birmingham.

Given $px^{-1}\dot{x}+ry^{-1}\dot{y}=y^{-n}x^{m}\dot{x}\div a$.

Multiply by y^{n} and it becomes

$$py^{n}x^{-1}\dot{x}+ry^{n-1}\dot{y}=x^{m}\dot{x}\div a:$$

assume $x^{q}y^{n}=sx^{v}\div av$; put this into fluxions, and

and $qx^{q-1}\dot{x}y^{n}+ny^{n-1}\dot{y}x^{q}=sx^{v-1}\dot{x}\div a$;

multiply this last expression by $r\div nx^{q}$, then

$rqy^{n}\dot{x}\div nx+ry^{n-1}\dot{y}=rsx^{v-q-1}\dot{x}\div na$; comp. this with

$py^{n}\dot{x}\div x+ry^{n-1}\dot{y}=x^{m}\dot{x}\div a$; to make them the same, then $rq\div n=p$, or $q=np\div r$; $rs\div n=1$, or $s=n\div r$; and $v-q-1=m$, or $v=m+q+1=m+(np\div r)+1$.

Whence, the relation of x and y will be expressed by $x^{(np\div r)}y^{n}=nx^{m+(np\div r)+1}\div ra(m+(np\div r)+1)$.

Observation 1. By a similar method of proceeding, we may find the relation of the fluents in the equation

$$ry^{-1}\dot{y}-px^{-1}\dot{x}=y^{-n}x^{m}\dot{x}\div a;$$

and in this case it will be found that

$$y^{n}\div x^{(np\div r)}=nx^{m-(np\div r)+1}\div ra\cdot(m-(np\div r)+1).$$

Obs. 2. From what is done above, it appears that the relation of x and y in the equation $ry^{-1}\dot{y}\pm px^{-1}\dot{x}=y^{-n}x^{m}\dot{x}\div a$, will always be expressed by the equation

$$y^{n}x^{\pm(np\div r)}=nx^{m\pm(np\div r)+1}\div ra\cdot(m\pm(n\div pr)+1).$$

Hence, suppose that when $x=a$, y is also $=a$; and the correct equation of the fluents become

$$y^{n}x^{\pm(np\div r)}=\left((rm\pm np+r)\cdot a^{n\pm(np\div r)+1}-na^{m\pm(np\div r)+1}+nx^{m\pm(np\div r)+1}\right)\div a\cdot(rm\pm np+r).$$

Obs.

Obſ. 3. If in the laſt equation we ſuppoſe $p=r=1$; $y^n x^{\pm n} = \left((m \pm n+1)\cdot a^{m \pm n+1} - na^{m \pm n+1} + \dot{n}x^{m \pm n+1}\right) \div a\cdot(m \pm n+1)$; and this agrees with queſtions 963, 502, *Ladie's Diary*, both being corrected, ſo that $x=y=a$.

Obſ. 4. If we ſuppoſe $p=2$, $r=3$, and $m=n=3$; the given equation becomes $3y^{-1}\dot{y} \pm 2x^{-1}\dot{x} = y^{-3}x^3\dot{x} \div a$, and the correct equation of the fluents will be

$$y^3 x^{\pm 2} = \left((9 \pm 6)\cdot a^{3 \pm 2+1} + 3x^{3 \pm 2+1}\right) \div a(9 \pm 6+3);$$

and this when the affirmative ſign is ſuppoſed to take place, becomes $y^3 x^2 = (x^6 + 5a^6) \div 6a$, which agrees with Mr. *Trott's* ſolution to queſtion 63, *Turner's* Exerciſes.

The fluents were alſo rightly determined by Mr. Ralph Simpſon, of Sunderland-bridge.

XIX. QUESTION 27, *anſwered by Mr.* Burdon.

Put $x-y=a$, and $(x-y)^{z^v}$ or $a^{z^v}=f$; alſo let the *hyp. log.* of a, f, and z be repreſented A, F and Z;

then will $\quad \dot{F} = z^v \dot{A} + A \times$ flux. z^v:

but fluxion $\quad z^v = vz^{v-1}\dot{z} + Zz^v\dot{v}$;

therefore $\quad \dot{F} = z^v\dot{A} + (vz^{v-1}\dot{z} + Zz^v\dot{v})\,A$;

but, $\quad \dot{A} = \dot{a} \div a = (\dot{x}-\dot{y}) \div (x-y)$ and $\dot{F} = \dot{f} \div f$;

hence, $\dot{F} = \dot{f} \div f = z^v\cdot(\dot{x}-\dot{y}) \div (x-y) + (vz^{v-1}\dot{z} + Zz^v\dot{v})\,A$;

or $\dot{f} = z^v\cdot(\dot{x}-\dot{y})\cdot(x-y)^{z^v-1} + (vz^{v-1}\dot{z} + Zz^v\dot{v})\cdot(x-y)^{z^v}\cdot A$,

and this is the fluxion required.

 Remark.

Remark. Mr. Richard Elliot *of* Liverpool, after finding the fluxion the ſame as Mr. Burdon, ſays, " if y be ſuppoſed $=0$; and $z=1$, the fluxion will then be $x^z\dot{z}\cdot hyp.\ log.\ x+zx^{z-1}\dot{x}$," which is the ſame as determined by Mr. Simpſon, at Art. 250, of his Fluxions.

The ſame by Mr. Jonathan Mabbot.

The fluxion of $(x-y)^{z^v}$, conſidering the exponent as conſtant is $=z^v\cdot(\dot{x}-\dot{y})\cdot(x-y)^{z^v-1}$; and the fluxion of the exponent (z^v) is $=vz^{v-1}\dot{z}+z^v\dot{v}\times$ *hyp. log.* z. Therefore the fluxion of $(x-y)^{z^v}$ will be $=z^v\cdot(\dot{x}-\dot{y})\cdot(x-y)^{z^v-1}+(vz^{v-1}\dot{z}+z^v\dot{v}\times$ *hyp. log.* $z)\cdot(x-y)^{z^v}\times$ *hyp. log.* $(x-y)$.

Otherwiſe by Mr. John Surtees, *of* Sunderland.

This gentleman from *page* 15, *Emerſon's Fluxions*, determines the fluxions the ſame as is done above, and then adds that " the ſame may be done without logarithms;" thus the fluxion

$$(x-y)^{z^v}=(x+\dot{x}-y-\dot{y})^{(z+\dot{z})^{v+\dot{v}}}-(x-y)^{z^v};$$

and when the firſt term is actually raiſed, and all the inſignificant quantities rejected;

$$(x-y)^{z^{v+\dot{v}}+vz^{v+\dot{v}-1}\dot{z}}+(z^{v+\dot{v}}$$

$$+vz^{v+\dot{v}-1}\dot{z})\times(x-y)^{z^{v+\dot{v}}+vz^{v+\dot{v}-1}\dot{z}-1}\times(\dot{x}-\dot{y})$$

$(\dot{x}-\dot{y})-(x-y^{z^{v}}$ will be the fluxion required.

This queſtion was alſo anſwered by Meſſrs. Lowry *and* Simpſon.

XX. QUESTION 28, *will be anſwered in* No. IV.

ARTICLE XXXI.

Four Propoſitions from *Lawſon.*

(To be anſwered in Number V.)

PROP. XVIII.

Let ABC be a triangle inſcribed in a circle, whoſe ſides AB and AC are equal, and from A any line be drawn meeting the circle again in D and BC in E; I ſay that the rectangle DAE is equal to the ſquare of AB.

PROP. XIX.

Things remaining as in the laſt propoſition, of lines touching the circle in A and C be drawn to meet in F and FD be drawn cutting BC in G; I ſay that the rectangle BCG is equal to the ſquare of CE.

PROP. XX.

Let ABC be a triangle inſcribed in a circle, whoſe ſides AB and AC are equal, and let AD be parallel to BC, and taking any point D therein, let the rectangle under AD and P be equal to the ſquare of AB or AC, and from the points A and D, let the

lines

lines AE, DE be inflected to any point E in the circle meeting BC in F and G; I say the rectangle under FG and P is equal to the rectangle BFC.

PROP. XXI.

If in AB the diameter of a circle be taken two points C and D, such, that AC : CB :: AD : DB, and D be within the circle, and DE be perpendicular to AB meeting the circle in E and F, and if through C any line be drawn meeting the circle in G and H, and the line DE in K, and GL touch the circle in G and meet DE in L; then I say the rectangle LDK is equal to the square of DE.

ARTICLE XXXII.

Two Propositions from *Stewart's* Theorems.

(To be answered in Number V.)

PROP. XIX. THEO. XVI.

Let there be any number of right lines given by position intersecting each other in a point, and let *a*, *b*, *c*, &c. be given magnitudes as many in number as there are right lines given by position; two right lines may be found that will be given by position, such, that if from any point there be drawn perpendiculars to the right lines given by position, and likewise there be drawn perpendiculars to the two right lines found, the square of the perpendicular drawn to one of the lines given by position, together with the space to which the square of the perpendicular drawn to another of the lines given by

by poſition has the ſame ratio that *a* has to *b*, together with the ſpace to which the ſquare of the perpendicular drawn to another of the lines given by poſition has the ſame ratio that *a* has to *c*, and ſo on, will be equal to the ſpace to which the ſum of the ſquares of the perpendiculars drawn to the two lines found has the ſame ratio that twice *a* has to the ſum of *a*, *b*, *c*, &c.

PROP. XX. THEO. XVII.

Let there be any number of right lines given by poſition, that are neither all parallel, nor interſecting each other in one point, and let *a*, *c*, *c*, &c. be given magnitudes as many in number as there are right lines given by poſition; two right lines may be found that will be given by poſition, ſuch, that if from any point there be drawn perpendiculars to the right lines given by poſition, and likewiſe there be drawn perpendiculars to the two right lines found, the ſquare of the perpendicular drawn to one of the lines given by poſition, together with the ſpace to which the ſquare of the perpendicular drawn to another of the lines given by poſition has the ſame ratio that *a* has to *b*, together with the ſpace to which the ſquare of the perpendicular drawn to another of the lines given by poſition has the ſame ratio that *a* has to *c*, and ſo on, will be equal to the ſpace to which the ſum of the ſquares of the perpendiculars drawn to the two lines found has the ſame ratio that twice *a* has to the ſum of *a*, *b*, *c*, &c. together with a given ſpace.

ARTICLE

ARTICLE XXXIII.

MATHEMATICAL QUESTIONS,

(To be anſwered in Number V.)

I. QUESTION 49, *by Mr.* Richard Wood.

Given the difference of the heights of two mountains, the diſtance between their tops, and likewiſe the neareſt diſtance between the ſurface of the earth and the line connecting their tops:—*Query* the height of each mountain?

II. QUESTION 50, *by Mr.* Thomas Bulmer.

Being at ſea on the forenoon of April 30th, 1796; I took the altitude of the ſun: and two hours after I took another altitude, and found their difference to be 10°:—*Query* the latitude of the place (being north) and the true times when the obſervations were made?

III. QUESTION 51, *by Mr.* William Burdon.

Let ABCD be any quadrilateral figure; let the ſide AD be divided into any number of equal parts in E, F, G, &c. and its oppoſite ſide BC into the ſame number of equal parts in *e*, *f*, *g*, &c; and if AB, DC be divided in any ratio in K, H, and KH be joined: then I ſay the ſtraight line KH will cut each of the other lines *e*E, *f*F, *g*G, &c. in the ſame ratio.—Required the demonſtration?

IV. QUESTION 52, *by Mr.* John Harris.

What is the diameter of a globe of ſound dry oak, which, when immerſed in common water, one

third

third part of its ſurface ſhall remain above the water?

V. QUESTION 53, *by Mr.* Ralph Simpſon.

There is a ſugar loaf in form of a cone, the radius of whoſe baſe is 6 and its altitude 18 inches, ſuſpended by its vertex from the ceiling of a room, in which is a lighted candle; the neareſt diſtance between the top of the candle and the baſe of the cone is 10 feet, and the neareſt between the top of the candle and the ceiling of the room is $5\frac{1}{2}$ feet; I demand the area of the ſhadow made by the cone upon the ceiling?

VI. QUESTION 54, *by a* Columbian.

A globe whoſe diameter was 12 inches, had a hole cut through it of 6 inches ſquare:—*Query* the ſolidity of the piece cut out?

VII. QUESTION 55, *by Mr.* George Brown.

A ball perfectly elaſtic, being projected from the top of a tower 80 feet high, with the initial velocity of 1500 feet in a ſecond, and in a direction of 20° 5′ elevation above the plane of the horizon, is ſuppoſed to meet a plane which touches the earth at the bottom of the tower; it is required to find the velocity of the ball, the time it has been in motion, and its diſtance from the bottom of the tower, at the end of the firſt reflection?

VIII. QUESTION 56, *by Mr.* William Pearſon.

Lunardi in one of his ærial voyages, dropped from his balloon, a leaden bullet of one inch diameter, which fell upon the ſcaffolding of a building and forced its way through a block of ſound elm 4 inches

inches thick, and then fell to the ground, a diſtance of 60 feet, in one ſecond. Now ſuppoſing (according to Dr. *Hutton's* experiments) that a caſt-iron ball of 2 inches diameter, impinging perpendicularly on ſound elm with a velocity of 1500 feet per ſecond, will penetrate 13 inches in its ſubſtance; it is required to determine Lunardi's height above the earth?

IX. QUESTION 37, *by Mr.* John Dawes, *London, lately of Birmingham.*

Given the diſtance of the nonageſimal degree from the meridian 16° 44′; the ſum of the altitude thereof and the altitude of the culmen cœli 78° 25′; required the altitude of the nonageſimal degree?

X. QUESTION 58, *by Mr.* Dawes.

Given the ſuns declination 23° north, his azimuth from the ſouth 86° 51′ 30″, and the ſum of his altitude and the latitude of the place (north); 92° 4′ to find the hour and latitude of the place?

XI. QUESTION 59, *by Mr.* J. E. H. Wadſon.

Given the hour of the day 4^h 35′, the ſuns azimuth from the ſouth 83° 41′ and the ſum of his altitude (north declination,) and the latitude of the place (north) 111° 45′; to find the latitude?

XII. QUESTION 60, *by Mr.* Wadſon.

Given the latitude of the place 60½° (north), the hour of the day 5^h 30′ P. M. and the ſum of the ſuns declination (north) and his altitude 135°; required the altitude?

XIII.

XIII. QUESTION 61, *by Mr.* J. H. Swale.

To determine, geometrically, two lines whoſe atio ſhall be given, ſuch, that a given line being aken from each, the rectangle of the differences may e equal to a given ſquare?

XIV. QUESTION 62, *by Mr.* John Lowry.

Given in a plane triangle, the ſum of one ſide and its adjacent ſegment, and the difference between the other ſide and its adjacent ſegment of the baſe made by the line biſecting the vertical angle, and the ſum of the perpendicular and the ſaid biſecting line, to conſtruct it?

XV. QUESTION 63, *by Mr.* Lowry.

Given the perimeter, the rectangle of the ſides, and the diſtance between the vertex and the centre of the inſcribed circle of a plane triangle to conſtruct it?

XVI. QUESTION 64, *by Mr.* W. Peacock. *Surveyor*, Birmingham.

Given the difference between the ſegments of the baſe made by the perpendicular, the ſum of the ſquares of the ſides, and the area of a plane triangle to conſtruct it?

XVII. QUESTION 65, *by* Geometricus.

Given the ſum of the ſides, the difference of the angles at the baſe, and the ſum of the perpendicular and difference between the ſegments of the baſe made thereby, to conſtruct the plane triangle?

XVIII. QUESTION 66, *by Mr.* Louis Hill, *Rowley*.

In a plane triangle there is given the difference of the angles at the bafe, the fum of the difference between the fegments of the bafe made by the perpendicular and the fum of the fides, and the rectangle of the fides to conftruct it?

XIX. QUESTION 67, *by Mr.* George Sanderfon.

In the equation $x - \frac{x + n^2a^2x^{2n-1}}{n-1} = y$, what is the value of y when $x=o$; and n equal, greater or lefs than, $\frac{1}{2}$?

XX. QUESTION 68, *by Mr.* Ztrepmog.

1. Let AB, BC, be the arcs of two different circles, it is propofed to find the radii thereof, having given AC the fum of the two radii, and AB +BC an extreme value.

2. Let there be given AC the fum of the radii, and the area an extreme value to find the radii.

3. Let there be given AC, (AB being an arc of a parabola, BC an arc of a circle) AB+BC an extreme value, when area ABC is given to find the latus rectum and radius of the parabola and circle?

ARTICLE

Demonstrations to Dr. Stewart's *Propositions proposed in* ARTICLE XIX, *and also Demonstrations to those two Propositions that has been already proposed as Questions in this Work, viz. Qu.* 4 *and* 19.

PROP. IX. THEO. VI. (*Qu.* 4).

Demonstrated by Mr. John Lowry.

LET there be any number of given points A, B, C, &c. (fig. 91, 92.) a point X may be found, such, that if from A, B, C, &c. there be drawn right lines to any point Y, and to the point X found, and if YX be joined, the sum of the squares of AY, BY, CY, &c. may be equal to the sum of the squares of AX, BX, CX, &c. together with the multiple by the number of the given points of the square of XY.

Suppose the number of given points to be three.

Join AB and bisect it in P; join PC and take PX equal to a third part of PC, and X will be the point required.

Join AX, BX, AY, BY, CX and PY; then (Prop. II.) the sum of the squares of AY, BY is equal to twice the sum of the squares AP, PY, and the square of CY, together with twice the square of PY is equal to the rectangle PCX together with three times the square of XY: therefore the sum of the squares of AY, BY, CY is equal to twice the square of AP together with the rectangle PCX together with three times the square of XY.

Again (Prop. II.) the sum of the squares of AX, BX is equal to twice the sum of the squares of AP, PX; and the square of CX to both, and the sum

sum of the square of AX, BX, CX is equal to twice the sum of the squares of AP, PX together with the square of CX: but CX is equal to twice PX; therefore twice the square of PX together with the square of CX is equal to the rectangle PCX; and therefore the sum of the squares of AY, BY, CY is equal to the sum of the squares of AX, BX, CX together with three times (or the multiple by the number of given points A, B and C of) the square of XY.

Note 1. The above method may easily be extended to any number of given points; for, if the number of given points had been supposed four, then XD being joined, and XS taken equal to a fourth part of XD; S will be the point required.

Note 2. The point found is the centre of gravity of the given points.

Corollary added, by Mr. Lowry.

Let there be any number of circles given by position, a point X may be found, such, that tangents being drawn to the circles from that point and from any other point Y, and XY be joined, the sum of the squares of the tangents drawn from the point Y will be equal to the sum of the squares of the tangents drawn from the point X together with the multiple by the number of the given circles of the square of XY.

Let A, B, C, &c. (fig. 93, pl. 7.) be the centres of the given circles; find the point X as in this proposition for the given points A, B, C, &c. and it will be the point required. For, from X draw the tangents XG, XH, XI, &c. and from Y the tangents YD, YE, YF, &c. and draw AD, AG, BF, BI, CE, CH, &c. from the centres to the points of contact.

Because

Becauſe the angles D, G, F, I, E, H are right ones, the ſum of the ſquares of DY, AD is equal to the ſquare of AY, the ſum of the ſquares of YF, BF is equal to the ſquare of BY, the ſum of the ſquares of YE, CE is equal to the ſquare of CY, and ſo on; and therefore the ſum of the ſquares of the tangents YD, YE, YF, &c. together with the ſum of the ſquares of the ſemi-diameters of the circles is equal to the ſum of the ſquares of AY, BY, CY, &c. The ſame way it is ſhewn that the ſum of the ſquares of the tangents XG, XH, XI, &c. together with the ſum of the ſquares of the ſemi-diameters of the circles is equal to the ſum of the ſquares of AX, BX, CX, &c. therefore the difference between the ſum of the ſquares of AY, BY, CY, &c. and the ſum of the ſquares of AX, BX, CX, &c. is equal to the difference between the ſum of the ſquares of the tangents YD, YE, YF, &c. and the ſum of the ſquares of the tangents XG, XH, XI, &c. But, (by this Prop.) the ſum of the ſquares of AY, BY, CY, &c. is equal to the ſum of the ſquares of AX, BX, CX, &c. together with the multiple by the number of given points of the ſquare of XY; therefore the ſum of the ſquares of the tangents YD, YE, YF, &c. is equal to the ſum of the ſquares of the tangents XG. XH, XI, &c. together with the multiple by the number of given circles of the ſquare of XY.

PROP. A. THEO.

Added by Mr. Lowry.

If from any number of given points right lines be drawn to one point ſuch, that the ſum of their ſquares may be equal to a given ſpace; the point of concourſe will fall in the circumference of a circle given by poſition.

Let A, B, C, &c. (fig. 92.) be the given points, which the right lines AY, BY, CY, &c. are drawn to another point Y ſuch, that the ſum of the ſquares of AY, BY, CY, &c. may be equal to a given ſpace; the point Y will fall in the circumference of a circle given by poſition.

Let the point X be found as in the laſt propoſition; then the ſum of the ſquares of AY, BY, CY, &c. is equal to the ſum of the ſquares of AX, BX, CX, &c. together with the multiple by the number of given points of the ſquare of XY. But the ſum of the ſquares of AY, BY, CY, &c. is given (by hypotheſis), and the ſum of the ſquares of AX, BX, CX, &c. is given, becauſe X is a given point; therefore the multiple by the number of given points of the ſquare of XY is given; and therefore XY itſelf is given in magnitude: but the point X is given; and therefore the point Y falls in the circumference of a circle given by poſition.

Cor. I. If tangents be drawn to any number of circles given by poſition to meet in one point, ſuch, that the ſum of their ſquares may be equal to a given ſpace; the point of concourſe of thoſe tangents will fall in the circumference of a circle given by poſition.

Let the ſum of the ſquares of YD, YE, YF, &c. (fig. 93, pl. 7.) be equal to a given ſpace; then the ſum of the ſquares of AY, BY, CY, &c. is given, equal to the ſum of the ſquares of YD, YE, YF, &c. together with the ſum of the ſquares of the ſemi-diameters of the circles; and therefore the point Y falls in the circumference of a circle given by poſition.

Cor. II. If right lines be drawn from any number of given points to one point, and from that point tangents be drawn to any number of circles given by poſition ſuch, that the ſum of the ſquares of the lines drawn from the given points together with the

the ſum of the ſquares of the tangents may be equal to a given ſpace, the *locus* of the point of concourſe will be a circle given by poſition.

Cor. III. If right lines be drawn from any number of given points to one point, ſuch, that the difference between the ſum of the ſquares of all the lines except one and the multiple of the ſquare of that one by the number of given points leſs one be equal to a given ſpace; the *locus* of the point of concourſe will be a right line given by poſition.

PROP. B. THEO.

Added by Mr. Lowry.

If from A (fig. 94. pl. 7,) the vertex of any triangle ABC there be drawn AD to any point D in the baſe, and if from D there be taken any equal diſtances DH, DL: and HF, LG, be drawn parallel to DA meeting AC, AB in F, G; and EQ, GR be drawn parallel to BC, meeting AD in Q, R; the ſum of the rectangles CAF, BAG will be equal to the ſum of the rectangle CDH, BDL, DAQ, DAR.

Draw AP perpendicular to CB. Becauſe of the parallels, AC is to AF as DC to DH, and as AD to AQ, the ſquare of AC will be to the rectangle CAF as the ſquare of DC to the rectangle CDH, and as the ſquare of AD to the rectangle DAQ, and, as twice the rectangle CDP to twice the rectangle HDP; therefore the ſquare of AC is to the rectangle CAF as the ſum of the ſquares of DC, AD, together with twice the rectangle CDP is to the ſum of the rectangles CDH, DAQ, together with twice the rectangle HDP: but, the ſquare of AC is equal to the ſum of the ſquares of DC, AD, together with twice the rectangle CDP; and therefore the rectangle CAF is equal to the ſum of the rectangles

CDH,

CDH, DAQ, together with twice the rectangle HDP. The same way it is shewn that the rectangle BAG is equal to the difference between the sum of the rectangles BDL, DAR and twice the rectangle LDP: but DH is equal to DL; and therefore twice the rectangle HDP is equal to twice the rectangle LDP; therefore the sum of the rectangles CAF, BAG, is equal to the sum of the rectangles CDH, BDL, DAQ, DAR.

PROP. C. THEO. Fig. 94, 95. Plate 7.

Added by Mr. Lowry.

In the right line CB, let any point D, between the points C, B, be taken, and from the points B, C, D, let there be drawn right lines to any point A, and DE be any given line; the space to which the square of CA has the same ratio that DE has to DB together with the space to which the square of BA has the same ratio that DE has to DC will be equal to the space to which the square of DC has the same ratio that DE has to DB together with the space to which the square of DB has the same ratio that DE has to DC together with the space to which the square of AD has the same ratio that DE has to CB.

*First.** When the point A is not in the line CB (fig. 94.)

Take AF to AC as BD to DE, and AG to AB as DC to DE; compleat the parallelograms DHFQ, DLGR. Because the rectangle CAF is to the square of AC as DB to DE, the rectangle CAF will be the space to which the square of AC has the same ratio that DE has to DB. The same way it is shewn, that the rectangle BAG is the space to which the square

* *This is Lemma 10th, Lib. 2. Simson. Loc. Plan. Apoll. of which Dr. Stewart's second Proposition is only a particular case.*

square of BA has the same ratio that DE has to DC, that the rectangle CDH is the space which the square of DC has the same ratio that DE has to DB, and that the rectangle BDL is the space to which the square of DB has the same ratio that DE has to DC. Again the rectangle DAQ is to the square of DA as AQ to DA, that is, as DB to DE, and the rectangle DAR is to the square of DA as AR to DA, that is, as DC to DE; therefore the sum of the rectangles DAQ, DAR is to the square of DA as CB to DE; and therefore the sum of the rectangles DAQ, DAR is the space to which the square of DA has the same ratio that DE has to CB. But, DL is to DB as AG to AB or DC to DE, and by permutation DL is to DC as DB to DE, that is, as AF to AC, that is, as DH to DC; therefore DL is equal to DH: and therefore the sum of the rectangles CAF, BAG is equal to the sum of the rectangles CDH, BDL, DAQ, DAR (by Prop. B.) or the space to which the square of CA has the same ratio that DE has to DB together with the space to which the square of BA has the same ratio that DE has to DC is equal to the space to which the square of DC has the same ratio that DE has to DB together with the space to which the square of DB has the same ratio that DE has to DC together with the space to which the square of AD has the same ratio that DE has to CB.

Second. When the point A is in the line CB (fig. 95. pl. 7.)

By the second part of Prop. II. the square of CA together with the space to which the square of BA has the same ratio that DB has to DC is equal to the rectangle BCD together with the space to which the square of AD has the same ratio that DB has to CB; therefore the space to which the square of CA has the same ratio that DE has to DB together with the spaoc to which the square of BA has the same ratio that

that DE has to DC is equal to the ſpace to which the rectangle BCD has the ſame ratio that DE has to DB together with the ſpace to which the ſquare of AD has the ſame ratio that DE has to CB. But, the rectangle BCD is equal to the ſquare of CD together with the rectangle CDB, and the ſpace to which the rectangle CDB has the ſame ratio that DE has to DB is equal to the ſpace to which the ſquare of DB has the ſame ratio that DE has to DC; therefore the ſpace to which the ſquare of CA has the ſame ratio that DE has to DB together with the ſpace to which the ſquare of BA has the ſame ratio that DE has to DC is equal to the ſpace to which the ſquare of DC has the ſame ratio that DE has to DB together with the ſpace to which the ſquare of DB has the ſame ratio that DE has to DC together with the ſpace to which the ſquare of AD has the ſame ratio that DE has to CB.

Note. The demonſtration of the ſecond part, contains alſo, a demonſtration of the firſt part.

PROP. X. THEO. VII. (Qu. 19.)

Demonſtrated by Mr. Lowry.

Let there be any number of given points A, B, C, &c. and let a, b, c, &c. be given magnitudes as many in number as there are given points; a point X may be found, ſuch, that if from the given points A, B, C, &c. there be drawn right lines to the point X, and from the given points and the point X there be drawn right lines to any point Y, the ſquare of AY together with the ſpace to which the ſquare of BY has the ſame ratio that a has to b, together with the ſpace to which the ſquare of CY has the ſame ratio that a has to c, and ſo on, will be equal to the ſquare of AX together with the ſpace to which the

ſquare

ſquare of BX has the ſame ratio that a has to b, together with the ſpace to which the ſquare of CX has the ſame ratio that a has to c, and ſo on, together with the ſpace to which the ſquare of XY has the ſame ratio that a has to the ſum of a, b, c, &c.

Let the number of given points be three. Join AB (fig. 91, 92.) and take AP to PB as b to a; join PC and take PX to XC as c to the ſum of a, b; and X will be the point required.

Join AX, BX, AY, BY, CY, PX and PY; then (Prop. II.) the ſquare of AY together with the ſpace to which the ſquare of BY has the ſame ratio that a has to b, (PB has to AP) is equal to the rectangle BAP together with the ſpace to which the ſquare of PY has the ſame ratio that a has to the ſum of a, b; but (Prop. C.) the ſpace to which the ſquare of CY has the ſame ratio that a has to c together with the ſpace to which the ſquare of PY has the ſame ratio that a has to the ſum of a, b is equal to the ſpace to which the ſquare of PX has the ſame ratio that a has to the ſum of a, b together with the ſpace to which the ſquare of CX has the ſame ratio that a has to c together with the ſpace to which the ſquare of XY has the ſame ratio that a has to the ſum of a, b, c; therefore the ſquare of AY together with the ſpace to which the ſquare of BY has the ſame ratio that a has to b together with the ſpace to which the ſquare of CY has the ſame ratio that a has to c is equal to the rectangle BAP together with the ſpace to which the ſquare of PX has the ſame ratio that a has to the ſum of a, b together with the ſpace to which the ſquare of CX has the ſame ratio that a has to c together with the ſpace to which the ſquare of XY has the ſame ratio that a has to the ſum of a, b, c. The ſame way it is ſhewn that the ſquare of AX together with the ſpace to which the ſquare of BX has the ſame ratio that a has

has to b is equal to the rectangle BAP together with the space to which the square of PX has the same ratio that a has to the sum of a, b; and the space to which the square of CX has the same ratio that a has to c, to both; then the square of AX together with the space to which the square of BX has the same ratio that a has to b together with the space to which the square of CX has the same ratio that a has to c is equal to the rectangle BAP together with the space to which the square of PX has the same ratio that a has to the sum of a, b together with the space to which the square of CX has the same ratio that a has to c; and therefore the square of AY together with the space to which the square of BY has the same ratio that a has to b together with the space to which the square of CY has the same ratio that a has to c is equal to the square of AX together with the space to which the square of BX has the same ratio that a has to b together with the space to which the square of CX has the same ratio that a has to c together with the space to which the square of XY has the same ratio that a has to the sum of a, b, c.

The above may easily be extended to any number of points.

Note. The point found is the centre of gravity of weights proportional to the magnitudes a, b, c, &c. placed at the given points A, B, C, &c.

The same demonstrated by Dr. Small.

Let there be any number of given points A, B, C, &c. and let a, b, c, &c. be given magnitudes as many in number as there are given points, a point X may be found, such, that if from A, B, C, &c. there be drawn straight lines to any point D, and also to X the point found, and if from DX be joined,

joined, $a{\cdot}AD^2+b{\cdot}BD^2+c{\cdot}CD^2$, &c. $=a{\cdot}AX^2+b{\cdot}BX^2+c{\cdot}CX^2$, &c. $+(a+b+c)DX^2$. Let m be $=3$, (fig. 58, pl. 3). Suppose the point X found. Join DX; from the given points A, B, C, draw AE, BF, CG, perpendicular to DX, and join AX, BX, CX.

Since $a{\cdot}AD^2+b{\cdot}BD^2+c{\cdot}CD^2=a{\cdot}AX^2+b{\cdot}BX^2+c{\cdot}CX^2+(a+b+c)\,DX^2$; and
$a{\cdot}AD^2=a{\cdot}AX^2+a{\cdot}DX^2-2a{\cdot}DX{\cdot}XE$ and
$b{\cdot}BD^2=b{\cdot}BX^2+b{\cdot}DX^2+2b{\cdot}DX{\cdot}XF$ and
$c{\cdot}CD^2=c{\cdot}CX^2+c{\cdot}DX^2+2c{\cdot}DX{\cdot}XG$; or
$a{\cdot}AD^2+b{\cdot}BD^2+c{\cdot}CD^2=a{\cdot}AX^2+b{\cdot}BX^2+c{\cdot}CX^2+(a+b+c)DX^2+2DX(-a{\cdot}XE+b{\cdot}XF+c{\cdot}XG)$; $a{\cdot}XE$ must be equal, and in the opposite direction to $b{\cdot}XF+c{\cdot}XG$. This will be effected by the following construction. Join AB, and divide it in H, so that $b{\cdot}BH=a{\cdot}AH$; that is, make $AH:BH=b:a$, and join HC, and divide it in X, so that $HX:CX=c:a+b$; or $(a+b)\,HX=c{\cdot}CX$. Then X will be the point required.

From H draw to DX, the perpendicular HK.

Since $a{\cdot}AH=b{\cdot}BH$, we shall have $a{\cdot}EK=b{\cdot}FK$; and since $(a+b)\,HX=c{\cdot}CX$, we shall also have $(a+b)\,KX=c{\cdot}GX$. Therefore since

$b{\cdot}XF=b{\cdot}FK-b{\cdot}KX$, and

$c{\cdot}XG=(a+b)KX$, we shall have

$b{\cdot}XF+c{\cdot}XG=b{\cdot}FK+a{\cdot}KX$

$=a{\cdot}EK+a{\cdot}KX=a{\cdot}XE$, and

$2DX(-a{\cdot}XE+b{\cdot}XF+c{\cdot}XG)=0$; therefore $a{\cdot}AD^2+b{\cdot}BD^2+c{\cdot}CD^2=a{\cdot}AX^2+b{\cdot}BX^2+c{\cdot}CX^2+(a+b+c)\,DX^2$, or $AD^2+\frac{b}{a}BD^2+\frac{c}{a}CD^2$

$$=AX^2+\frac{h}{a}BX^2+\frac{c}{a}CX^2+\left(\frac{a+b+c}{a}\right)DX^2.$$

Cor. I. Demonſtrated by Mr. Lowry.

Let there be any number of circles given by poſition, and about every circle let an equilateral figure be deſcribed, a point may be found, ſuch, that if from any point Y there be drawn perpendiculars to all the ſides of the figures and a ſtraight line to the point found, twice the ſum of the ſquares of the perpendiculars will be equal to the multiple of the ſquare of the line drawn to the point found, by the number of the ſides of the figures together with a given ſpace.

Let A, B, C, &c. (fig. 96, pl. 7.) be the centres of the given circles, a the number of the ſides of the figure deſcribed about the circle whoſe centre is A, b the number of the ſides of the figure deſcribed about the circle whoſe centre is B, c the number of the ſides of the figure deſcribed about the circle whoſe centre is C, &c. Join the centres A, B, and take AP to PB as b to a; join PC and take PX to XC as c to the ſum of a, b, and ſo on; then P will be the required point for the two circles A, B and X the required point for the three circles A, B, C, &c.

Join AY, BY, CY, PY, XY and draw YG, YH, YI, YK, YD, YE, YF, YL, YM, YN, YO, YR, &c. perpendicular to the ſides of the figures; then (Prop. V.) twice the ſum of the ſquares of the perpendiculars YG, YH, YI, YK, &c. is equal to the multiple of the ſquare of AY by the number a, together with twice the multiple of the ſquare of the ſemi-diameter of the circle whoſe centre is A by the ſame number, and twice the ſum of the ſquares of YD, YE, YF, &c. is equal to the multiple of the ſquare of BY by the number b, together with twice the multiple of the ſquare of the ſemidiameter of the circle whoſe centre is B by th[illegible]me number, and the ſum of the ſquares of [illegible]M, YN, YO, YR, &c. is equal to the multiple

ple of the ſquare of CY by the number c, together ith twice the multiple of the ſquare of the ſemi-ameter of the circle whoſe centre is C by the me number; therefore twice the ſum of the ſquares the perpendiculars is equal to twice the multiple the ſquares of the ſemi-diameters by the number the ſides of the figures deſcribed about each circle ſpectively, together with the multiples of AY, Y, CY, &c. by the ſame a, b, c, &c. reſpectively. ut (by this Prop.) the ſquare of AY, together ith the ſpace to which the ſquare of BY as the ſame ratio that a has to b, together with e ſpace to which the ſquare of CY has the ſame tio that a has to c, &c. is equal to the ſquare of X, together with the ſpace to which the ſquare f BX has the ſame ratio that a has to b, together ith the ſpace to which the ſquare of CX has the me ratio that a has to c, &c. together with the pace to which the ſquare of XY has the ſame atio that a has to the ſum of a, b, c, &c. and herefore the multiple of the ſquare of AY by the umber a, together with the multiple of the ſquare f BY by the number b, together with the mul-iple of the ſquare of CY by the number c, and o on, is equal to the multiple of the ſquares of AX, BX, CX, &c. by the ſame numbers reſpectively, ogether with the multiple of the ſquare of XY y the ſum of a, b, c, &c. therefore twice the ſum f the ſquares of the perpendiculars is equal to twice he multiples of the ſum of the ſquares of the ſemi-liameters of the given circles, by the number of he ſides of the figures deſcribed about each circle eſpectively, together with the multiples of the ſquares of AX, BX, CX, &c. by the ſame num-bers reſpectively, together with the multiple of the ſquare of XY by the ſum of a, b, c, &c. But wice the multiple of the ſum of the ſquares of he ſemi-diameters of the given circles by the num-

 ber

ber of the ſides of the figures, deſcribed about each circle reſpectively, together with the multiples of the ſquares of AX, BX, CX, &c. by the ſame numbers reſpectively is a given ſpace; therefore twice the ſum of the ſquares of the perpendiculars is equal to the multiple of the ſquare of XY by the number of the ſides of the figures, together with a given ſpace.

The ſame Demonſtrated by Dr. Small.

Let the number of circles given by poſition be two; let a be the number of the ſides of the figure deſcribed about the circle whoſe centre is A, b the number of the ſides of the figure deſcribed about the circle whoſe centre is B, CD, CE, CF, the perpendiculars to the ſides of the firſt figure, and CG, CH, CK, CL, the perpendiculars to the ſides of the ſecond, (fig. 38, pl. 2.). Join the centres A, B, and divide AB in X, ſo that $AX : BX = b : a$, X will be the point required. $2(CD^2+CE^2+CF^2)=2a\cdot AM^2+a\cdot AC^2$ (Theo. 3.) In like manner $2(CG^2+CH^2+CK^2+CL^2)=2b\cdot BN^2+b\cdot BC^2$. Therefore $2(CD^2+CE^2+CF^2+CG^2+CH^2+CK^2+CL^2) = 2a\cdot AM^2 + 2b\cdot BN^2+a\cdot AC^2+b\cdot BC^2$. But, $a\cdot AC^2+b\cdot BC^2=(a+b)AX\cdot BX+(a+b)CX^2$ (Prop. I.) and $2a\cdot AM^2+2b\cdot BN^2+(a+b)AX\cdot BX$ are given ſpaces. Therefore $2(CD^2+CE^2+CF^2+CG^2+CH^2+CK^2+CL^2) = (a+b)\,CX^2 + A^2$, A^2 being a given ſpace.

Cor. II. Demonſtrated by Mr. Lowry.

Let any number of ſemi-circles be given by poſition, and let an equilateral figure be deſcribed about every ſemi-circle, a point may be found, ſuch, that if from any point there be drawn perpendiculars to all the ſides of the figures, and a ſtraight line to the point found, twice the ſum of the ſquares of the perpendiculars will be equal to the multiple of the ſquare of the line drawn to the point

point found, by the number of all the ſides of the figures, together with a given ſpace. Let A, B, &c. (fig. 97. pl. 7.) be the centres of the given ſemi-circles, *a* the number of the ſides of the figure deſcribed about the ſemi-circle whoſe centre is A, *b* the number of the ſides of the figure deſcribed about the ſemi-circle whoſe centre is B, &c. Find the points L, T, as in Prop. VIII. join LT and take LX to XT as *b* to *a*, and X will be the point required for the two ſemi-circles A, B.

Let Y be any other point, YD, YE, YF, &c. the perpendiculars to the ſides of the figure deſcribed about the ſemi-circle whoſe centre is A, and YG, YH, YI, YR, &c. the perpendiculars to the ſides of the figure deſcribed about the ſemi-circle whoſe centre is B, &c.

Join LY, TY, &c. and XY; then (Prop. VIII.) twice the ſum of the ſquares of YD, YE, YF, &c. is equal to the multiple of the ſquare of YL by the number *a*, together with the multiple of the rectangle KAL by the ſame number, and twice the ſum of the ſquares of YG, YH, YI, YR, &c. is equal to the multiple of the ſquare of YT by the number *b*, together with the multiple of the rectangle SBT by the ſame number; therefore twice the ſum of the ſquares of the perpendiculars is equal to the multiple of the ſquare of YL by the number *a*, together with the multiple of the ſquare of YT by the number *b*, &c. together with the multiples of the rectangles KAL, SBT, by the ſame numbers *a*, *b*, &c. reſpectively. Again, it is ſhewn in the ſame way, as in *Cor.* 1. that the multiple of the ſquare of YL by the number *a*, together with the multiple of the ſquare of YT by the number *b*, &c. is equal to the multiple of the ſquare of LX by the number *a*, together with the multiple of the ſquare of TX by the number *b*, &c. together with the multiple of the ſquare of XY by the

ſum

um of the numbers a, b, &c. and therefore twice the fum of the fquares of the perpendiculars is equal to the multiple of the rectangle KAL by the number a, together with the multiple of the rectangle SBT by the number b, &c. together with the multiples of the fquares of LX, TX, &c. by the fame numbers a, b, &c. refpectively, together with the multiple of the fquare of XY by the fum of the numbers a, b, &c. But the multiple of the rectangle KAL by the number a, together with the multiple of the rectangle SBT by the number b, &c. together with the multiples of the fquares of LX, TX, &c. by the fame numbers a, b, &c. refpectively, is a given fpace. Therefore twice the fum of the fquares of the perpendiculars is equal to the multiple of the fquare of XY by the number of the fides of the figures, together with a given fpace.

Cor. III. *Demonftrated by Mr.* Lowry.

Let there be any number of circles and femi-circles given by pofition, and about every circle, and femi-circle, let an equilateral figure be defcribed, a point may be found, fuch, that if from any point there be drawn perpendiculars to all the fides of the figures, and a ftraight line to the point found, twice the fum of the fquares of the perpendiculars will be equal to the multiple of the fquare of the line drawn to the point found, by the number of the fides of the figures, together with a given fpace.

In addition to the figure to the laft corollary, let O be the centre of a circle given by pofition, and c the number of the fides of the figure defcribed about it. Join OX and take XP to PO as c to the fum of a, b; then will P be the point required for the circle O, and the two femi-circles A, B.

Join OY, PY, and draw YM, YN, YQ, &c. perpendicular to the fides of the figure defcribed about

about the circle. Then (Cor. II.) twice the ſum of the ſquares of the perpendiculars to the ſides of the figures deſcribed about the ſemi-circles, is equal to the multiple of the ſquare of XY by the number of the ſides, together with a given ſpace, and (Prop. V.) twice the ſum of the ſquares of the perpendiculars YM, YN, YQ, &c. is equal to the multiple of the ſquare of OY by the number c, together with a given ſpace. But the multiple of the ſquare of XY by the ſum of a, b, together with the multiple of the ſquare of OY by the number c, is equal to the multiple of the ſquare of PX by the ſum of a, b, together with the multiple of the ſquare of PO by the number c, together with the multiple of the ſquare of PY by the ſum of a, b, c, that is, equal to a given ſpace, together with the multiple of the ſquare of PY by the ſum of a, b, c; therefore twice the ſum of the ſquares of the perpendiculars is equal to the multiple of the ſquare of PY by the number of the ſides of the figures, together with a given ſpace.

Cor. IV. Added by Mr. Lowry.

Let any number of ſegments of circles be given by poſition, and let an equilateral figure be deſcribed about every ſegment, a point may be found, ſuch, that if from any point there be drawn right lines to all the points of contact made by the circumſcribing figures, and alſo to the point found, the ſum of the ſquares of the lines drawn to the points of contact will be equal to the multiple of the ſquare of the line drawn to the point found, by the number of the ſides of the figures, together with a given ſpace.

Cor.

Cor. V. Added by Mr. Lowry.

Let any number of circles be given by poſition, and let an equilateral figure be inſcribed in every circle, a point may be found, ſuch; that if from any point right lines be drawn to all the angles of the figures, and alſo to the point found, the ſum of the ſquares of the lines drawn to the angles of the figures will be equal to the multiple of the ſquare of the line drawn to the point found by the number of the ſides of the figures, together with a given ſpace.

Cor. VI. Added by Mr. Lowry.

Let any number of ſemi-circles be given by poſition, and let an equilateral figure be inſcribed in every ſemi-circle, a point may be found, ſuch, that if from any point there be drawn perpendiculars to all the ſides of the figures, and a right line to the point found, twice the ſum of the ſquares of the perpendiculars will be equal to the multiple of the ſquare of the line drawn to the point found by the number of the ſides of the figures, together with a given ſpace.

Cor. VII. Added by Mr. Lowry.

Let there be any number of circles and ſegments of circles given by poſition, and let an equilateral figure be deſcribed about every ſegment, and alſo, let an equilateral figure be inſcribed in every circle, a point may be found, ſuch, that if from any point there be drawn right lines to all the points of contact made by the circumſcribing figures, and alſo to all the angles of the inſcribed figures, and to the point found, the ſum of the ſquares of the lines drawn to the points of contact of the circumſcribed figures, together with the ſum of the

ſquares

squares of the lines drawn to the angles of the inscribed figures will be equal to the multiple of the square of the line drawn to the point found by the number of the sides of the figures, together with a given space.

PROP. D. THEO*.

Added by Mr. Lowry.

If from any number of given points, right lines be drawn to one point, such, that the sum of the rectilineal figures given in species described upon them, may be equal to a given space; their point of concourse will fall in the circumference of a circle given by position.

Let the points A, B, C, &c. (fig. 92, pl. 7.) be given, from whence the right lines AY, BY, CY, &c. are drawn to the point Y, such, that the sum of the rectilineal figures given in species described upon them, may be equal to a given space.

Because the figures are given in species, their ratios to the squares of the lines upon which they are described will be given. Let the ratio of the figure described upon AY, be to the square of AY as *a* to *m*, and the ratio of the figure described upon BY, be to the square of BY as *b* to *m*, and the ratio of the figure described upon CY, be to the square of CY as *c* to *m*, &c. and let the point X be found (by Prop. X.) for the given points A, B, C, &c. and the given magnitudes *a*, *b*, *c*, &c. Join AX, BX, CX, &c. and XY; then (Prop. X.) the square of AY, together with the space to which the square of BY has the same ratio that *a* has to *b*, together with the space to

* This Prop. and Prop. A, contains the whole of the Fifth Prop. of the Second Book of *Apollonius* on Plani Loci.

which

which the ſquare of CY has the ſame ratio that *a* has to *c*, &c. is equal to the ſquare of AX, together with the ſpace to which the ſquare of BX has the ſame ratio that *a* has to *b*, together with the ſpace to which the ſquare of CX has the ſame ratio that *a* has to *c*, &c. together with the ſpace to which the ſquare of XY has the ſame ratio that *a* has to the ſum of *a*, *b*, *c*, &c. therefore the ſpace to which the ſquare of AY has the ſame ratio that *m* has to *a*, together with the ſpace to which the ſquare of BY has the ſame ratio that *m* has to *b*, together with the ſpace to which the ſquare of CY has the ſame ratio that *m* has to *c*, &c. is equal to the ſquare of AX, together with the ſpace to which the ſquare of BX has the ſame ratio that *m* has to *b*, together with the ſpace to which the ſquare of CX has the ſame ratio that *m* has to *c*, &c. together with the ſpace to which the ſquare of XY has the ſame ratio that *m* has to the ſum of *a*, *b*, *c*, &c. that is, the ſum of the rectilineal figures deſcribed upon AY, BY, CY, &c. is equal to the ſimilar figures deſcribed upon AX, BX, CX, &c. together with the ſpace to which the ſquare of XY has the the ſame ratio that *m* has to the ſum of *a*, *b*, *c*, &c. But the ſum of the figures deſcribed upon AX, BX, CX, &c. is equal to a given ſpace, becauſe the lines AX, BX, CX, &c. are given, and the figures deſcribed upon them are given in ſpecies; therefore the ſpace to which the ſquare of XY has the ſame ratio that *m* has to the ſum of *a*, *b*, *c*, &c. is given; and therefore XY itſelf is given in magnitude; but the point X is given; therefore the point Y falls in the circumference of a circle given by poſition.

Cor. 1. If from any number of given points, right lines be drawn to one point, ſuch, that the ſum of the areas of the circles deſcribed upon them

as diameters may be equal to a given ſpace; their point of concourſe will fall in the circumference of a circle given by poſition.

Cor. 2. Let any number of circles be given by poſition, and let an equilateral figure be deſcribed about every circle, and perpendiculars be drawn to the ſides of the figures to meet in one point, ſuch, that the ſum of their ſquares may be equal to a given ſpace; the *locus* of that point will be a circle given by poſition.

Cor. 3. Let any number of ſemi-circles be given by poſition, and let an equilateral figure be deſcribed about every ſemi-circle, and if perpendiculars be drawn to all the ſides of the figures to meet in one point, ſuch, that the ſum of their ſquares may be equal to a given ſpace; the *locus* of that point will be a circle given by poſition.

Cor. 4. Let any number of circles and ſemi-circles be given by poſition, and let an equilateral figure be deſcribed about every circle and ſemi-circle, and if perpendiculars be drawn to all the ſides of the figures to meet in one point, ſuch, that the ſum of their ſquares may be equal to a given ſpace, the *locus* of that point will be a circle given by poſition.

Cor. 5. Let any number of ſegments of circles be given by poſition, and let an equilateral figure be deſcribed about every ſegment, and from all the points of contact let right lines be drawn to meet in one point, ſuch, that the ſum of their ſquares may be equal to a given ſpace; the *locus* of that point will be a circle given by poſition.

Cor. 6. Let any number of circles be given by poſition, and let an equilateral figure be inſcribed in every circle, then if right lines be drawn from all the angles of the figures to meet in one point, ſuch, that the ſum of their ſquares may be equal to a given ſpace; the *locus* of that point will be a circle given by poſition.

Cor. 7. Let there be any number of ſemi-circles given by poſition, and let an equilateral figure be inſcribed in every ſemi-circle, then if perpendiculars be drawn to all the ſides of the figures to meet in one point, ſuch, that the ſum of their ſquares may be equal to a given ſpace; the *locus* of that point will be a circle given by poſition.

Cor. 8. Let any number of circles and ſegments of circles be given by poſition, and let an equilateral figure be deſcribed about every ſegment, and alſo, let an equilateral figure be inſcribed in every circle, then if right lines be drawn from all the points of contact of the circumſcribed figure, and alſo from all the angles of the inſcribed figures to meet in one point, ſuch, that the ſum of their ſquares may be equal to a given ſpace; the *locus* of that point will be a circle given by poſition.

PROP. XI., THEO. VIII.

Demonſtrated by Mr. Lowry.

Let there be any number of given points A, B, C, &c. (fig. 98, pl. 7.) two points X, Y may be found, ſuch, that if from any point P there be drawn right lines to A, B, C, &c. and to X, Y the two points found, twice the ſum of the ſquares of AP, BP, CP, &c. will be equal to the multiple of the ſum of the ſquares of PX, PY by the number of given points. Find the point O as in Prop. IX. for the given points A, B, C, &c. and let a ſquare be found, whoſe multiple by the number of given points may be equal to the ſum of ſquares of AO, BO, CO, &c. with a diſtance equal to the ſide of this ſquare, and about the point O as a centre, let a circle be deſcribed, and the extremities X, Y of any diameter will be two ſuch points as are required.

By

By Prop. IX. the ſum of the ſquares of AP, BP, CP, &c. is equal to the ſum of the ſquares of AO, BO, CO, &c. together with the multiple by the number of given points of the ſquare of OP. And *(by conſtruction)* the ſum of the ſquares of AO, BO, CO, &c. is equal to the multiple by the number of the given points of the ſquare of the ſemi-diameter OX, (or OY); therefore twice the ſum of the ſquares of AP, BP, CP, &c. is equal to twice the multiple by the number of given points of the ſum of the ſquares of OX, OP. But, XY is biſected in O; therefore (Prop. II. *Cor.*) twice the ſum of the ſquares OX, OP, is equal to the ſum of the ſquares XP, YP; and therefore twice the ſum of the ſquares of AP, BP, CP, &c. is equal to the multiple by the number of the given points of the ſquares of PX, PY.

The ſame demonſtrated by Dr. Small.

Let there be any number, m, of given points A, B, C, &c. two points X, Y, may be found, ſuch, that if from any point D, ſtraight lines be drawn to A, B, C, &c. and to X, Y, $2(DA^2+DB^2+DG^2) = m\,(DX^2+DY^2)$. This propoſition follows directly from Theor. 6. Let $m=3$, and let E (fig. 39, pl. 2.) be the centre of gravity of the three points A, B, C. The ſquares of EA, EB, EC, are given, and conſequently a ſquare $=\frac{1}{3}(EA^2+EB^2+EC^2)$ may be found. On E with the diſtance EX equal to the ſide of this ſquare, deſcribe a circle. The extremities X, Y, of any diameter, will be two ſuch points as are required. For,

$$DA^2+DB^2+DC^2=EA^2+EB^2+EC^2+3\cdot ED^2, \text{ (Theor. 6.)}$$

But, $EA^2+EB^2+EC^2=3\cdot EX^2$, therefore

$$2\cdot(DA^2+DB^2+DC^2=6\,(EX^2+ED^2)$$
$$=3\,(DX^2+DY^2) \text{ (Prop. I.)}$$

Corollary added by Mr. Lowry.

Let there be any number of circles given by po-

ſition, two points may be found, ſuch, that if from any point tangents be drawn to the circles and right lines to the two points found, twice the ſum of the ſquares of the tangents will be equal to the multiple by the number of the given circles of the ſum of the ſquares of the lines drawn to the two points found.

PROP. XII. THEO. IX.

Demonſtrated by Mr. Lowry.

Let there be any number of given points A, B, C, &c (fig. 98, pl. 7.) and let a, b, c, &c. be given magnitudes as many in number as there are given points, two points X, Y may be found, ſuch, that if from any point P right lines be drawn to A, B, C, &c. and to X, Y, the two points found, the ſquare of AP together with the ſpace to which the ſquare of BP has the ſame ratio that a has to b together with the ſpace to which the ſquare of CP has the ſame ratio that a has to c and ſo on, will be equal to the ſpace to which the ſum of the ſquares of PX, PY, has the ſame ratio that twice a has to the ſum of a, b, c, &c.

Find the point O as in Prop. X. for the given points A, B, C, &c. and the given magnitudes a, b, c, &c. and let a ſquare be found equal to the ſpace to which the ſquare of AO together with the ſpace to which the ſquare of BO has the ſame ratio that a has to b together with the ſpace to which the ſquare of CO has the ſame ratio that a has to c, &c. has the ſame ratio that the ſum of a, b, c, &c. has to a.

Then with a diſtance equal to the ſide of this ſquare and centre O, let a circle be deſcribed; and the extremities X, Y of any diameter, will be two ſuch points as are required.

By Prop. X. the ſquare of AP together with the ſpace to which the ſquare of BP has the ſame ratio

that

that a has to b together with the ſpace to which the ſquare of CP has the ſame ratio that a has to c, &c. is equal to the ſquare of AO together with the ſpace to which the ſquare of BO has the ſame ratio that a has to b together with the ſpace to which the ſquare of CO has the ſame ratio that a has to c, &c. together with the ſpace to which the ſquare of OP has the ſame ratio that a has to the ſum of a, b, c, &c. And *(by conſtruction)* the ſquare of AO together with the ſpace to which the ſquare of BO has the ſame ratio that a has to b together with the ſpace to which the ſquare of CO has the ſame ratio that a has to c, &c. is equal to the ſpace to which the ſquare of the ſemi-diameter OX (or OY) has the ſame ratio that a has to the ſum of a, b, c, &c. therefore the ſquare of AP together with the ſpace to which the ſquare of BP has the ſame ratio that a has to b together with the ſpace to which the ſquare of CP has the ſame ratio that a has to c, &c. is equal to the ſpace to which the ſum of the ſquares of OP, OX has the ſame ratio that a has to the ſum of a, b, c, &c. that is, equal (becauſe the ſum of the ſquares of PX, PY is equal to twice the ſum of the ſquares of OP, OX, by *Cor.* to Prop. II.) to the ſpace to which the ſum of the ſquares of PX, PY has the ſame ratio that twice a has to the ſum of a, b, c, &c.

The ſame demonſtrated by Dr. Small.

Let there be any number, m, of given points A, B, C, &c. and let a, b, c, &c. be given magnitudes, as many in number as there are given points, two points X, Y, may be found, ſuch, that if from any point D there be drawn ſtraight lines to A, B, C, &c. and to X, Y.

$$DA^2+\frac{b}{a}DB^2+\frac{c}{a}DC^2 \text{ \&c.}=\left(\frac{a+b+c}{2a}\right)(DX^2+DY^2)$$

This Proposition follows, in the same manner from Theor. 7. as the last did from Theor. 6. Let $m=3$. Let E (fig. 39, plate 2.) be a point such that $DA^2+\frac{b}{a}DB^2+\frac{c}{a}DC^2=EA^2+\frac{b}{a}EB^2+\frac{c}{a}EC^2+\left(\frac{a+b+c}{a}\right)ED^2$. On E as a centre, with the distance $EX=\sqrt{\frac{a}{a+b+c}(EA^2+\frac{b}{a}EB^2+\frac{c}{a}EC^2)}$ describe a circle. The extremities X, Y, of any diameter will be two such points as are required. For

$$DA^2+\frac{b}{a}DB^2+\frac{c}{a}DC^2=EA^2+\frac{b}{a}EB^2+\frac{c}{a}EC^2+\left(\frac{a+b+c}{a}\right)ED^2,$$

and $EA^2+\frac{b}{a}EB^2+\frac{c}{a}EC^2=\left(\frac{a+b+c}{a}\right)EX^2$. Therefore

$$2(DA^2+\frac{b}{a}DB^2+\frac{c}{a}DC^2+)=2\left(\frac{a+b+c}{a}\right)(ED^2+EX^2)=$$

$$\left(\frac{a+b+c}{a}\right)(DX^2+DY^2),\text{ (Prop. I.) Or,}$$

$$DA^2+\frac{b}{a}DB^2+\frac{c}{a}DC^2=\left(\frac{a+b+c}{2a}\right)(DX^2+DY^2).$$

Cor. I. Added by Mr. Lowry.

Let there be any number of circles given by position, and let an equilateral figure be described about every circle; two points may be found, such, that if from any point there be drawn perpendiculars to all the sides of the figures, and right lines to the two points found, four times the sum of the squares of the perpendiculars will be equal to the multiple by the number of the sides of the figures of the sum of the squares of the lines drawn to the two points found.

Cor.

Cor. II. Added by Mr. Lowry.

Let there be any number of ſemi-circles given by poſition, and let an equilateral figure be deſcribed about every ſemi-circle, two points may be found, ſuch, that if from any point perpendiculars be drawn to all the ſides of the figures, and right lines to the two points found, four times the ſum of the ſquares of the perpendiculars will be equal to the multiple by the number of the ſides of the figures of the ſum of the ſquares of the lines drawn to the two points found.

Cor. III. Added by Mr. Lowry.

Let any number of circles and ſemi-circles be given by poſition, and about every circle and ſemi-circle let an equilateral figure be deſcribed, two points may be found, ſuch that if from any point perpendiculars be drawn to all the ſides of the figures, and right lines to the two points found, four times the ſum of the ſquares of the perpendiculars will be equal to the multiple by the number of the ſides of the figures of the ſum of the ſquares of the lines drawn to the two points found.

Cor. IV. Added by Mr. Lowry.

Let any number of ſegments of circles be given by poſition, and let an equilateral figure be deſcribed about every ſegment, two points may be found, ſuch, that if from any point there be drawn right lines to all the points of contact of the circumſcribed figures, and to the two points found, twice the ſum of the ſquares of the lines drawn to the points of contact will be equal to the multiple by the number of all the ſides of the figures of the ſum of the ſquares of the lines drawn to the two points found.

Cor. V. Added by Mr. Lowry.

Let any number of circles be given by position, and let an equilateral figure be inſcribed in every circle; two points may be found, ſuch, that if from any point there be drawn right lines to all the angles of the inſcribed figures, and to the two points found, twice the ſum of the ſquares of the lines drawn to all the angles of the inſcribed figures will be equal to the multiple by the number of the ſides of the figures of the ſum of the ſquares of the lines drawn to the two points found.

Cor. VI. Added by Mr. Lowry.

Let any number of ſemi-circles be given by poſition, and let an equilateral figure be inſcribed in every ſemi-circle; two points may be found, ſuch, that if from any point perpendiculars be drawn to all the ſides of the figure, and right lines to the two points found, four times the ſum of the ſquares of the perpendiculars will be equal to the multiple by the number of the ſides of the figures of the ſum of the ſquares of the lines drawn to the two points found.

Cor. VII. Added by Mr. Lowry.

Let any number of circles and ſegments of circles be given by poſition, and let an equilateral figure be deſcribed about every ſegment, and alſo, let an equilateral figure be inſcribed in every circle; two points may be found, ſuch, that if from any point, right lines be drawn to all the points of contact of the circumſcribed figures, and to all the angles of the inſcribed figures, and to the two points found, twice the ſum of the ſquares of the lines drawn to all the points of contact of the circumſcribed figures, and to all the angles of the inſcribed figures will be equal to the multiple by the number of all the ſides of the

figures

figures of the ſum of the ſquares of the lines drawn to the two points found.

PROP. XIII. THEO. X.

Demonſtrated by Mr. Lowry.

Let there be any number of right lines AL, BM, CN, &c. (fig. 99, pl. 7.) given by poſition, and parallel to each other; a right line XY may be found parallel to them, ſuch, that if from any point P there be drawn PABC, &c. perpendicular to the given lines, the ſum of the ſquares of AP, BP, CP, &c. will be equal to the multiple by the number of given lines of the ſquare of PX, together with a given ſpace. Find the point X as in Prop. IX. for the given points A, B, C, &c. through X draw the right line XY parallel to the given lines, and it will be the line required.

By Prop. IX. The ſum of the ſquares of AP, BP, CP, &c. is equal to the ſum of the ſquares of AX, BX, CX, &c. together with the multiple by the number of the given lines of the ſquare of PX.

But, the ſum of the ſquares of AX, BX, CX, &c. is a given ſpace; therefore the ſum of the ſquares of AP, BP, CP, &c. is equal to the multiple by the number of the given lines of the ſquare of PX together with a given ſpace.

The ſame demonſtrated by Dr. Small.

Let there be any number, *m*, of parallel ſtraight lines AB, CD, EF, &c. (fig. 40, plate 2.) given by poſition, a ſtraight line XY may be found parallel to them, ſuch, that if from any point G, perpendiculars GA, GC, GE, &c. be drawn to AB, CD, EF, &c. and the line GX perpendicular to XY,

GA²+

$GA^2+GC^2+GE^2$, &c. $=m\cdot GX^2+A^2$, A^2 being a given ſpace.

This propoſition is one of the ſimpleſt caſes of Theor. 6. A line XY parallel to AB, drawn through X, the centre of gravity of the points A, C, E, where a perpendicular from G, meets the parallels AB, CD, EF will be the line required. For, $GA^2+GC^2+GE^2=XA^2+XC^2+XE^2+3GX^2$ (Theor. 6.) and $XA^2+XC^2+XE^2$ is a given ſpace.

PROP. XIV. THEO. XI.

Demonſtrated by Mr. Lowry.

Let there be any number of right lines AB, AC, AD, &c. (fig. 100, plate 7.) interſecting each other in the point A, and making all the angles about the point A equal, and from any point P let the perpendiculars PB, PC, PD, &c. be drawn to AB, AC, AD, &c. and AP be joined, twice the ſum of the ſquares of PB, PC, PD, &c. will be equal to the multiple by the number of lines of the ſquare of AP.

Becauſe the angles at B, C, D, &c. are right, the points B, C, D, &c. will be in the circumference of a circle whoſe diameter is AP; therefore (Lem. II.) becauſe the circle paſſes through A, the circumference will be divided into equal parts in the points B, C, D, &c. as many in number as there are right lines AB, AC, AD, &c. and therefore (Prop. IV.) the ſum of the ſquares of PB, PC, PD, &c. will be equal to twice the multiple of the ſquare of the ſemi-diameter, or half AP, by the number of the given points A, B, C, &c. therefore the ſum of the ſquares of the perpendiculars PB, PC, PD, &c. is equal to the multiple of the ſquare of AP by the number of right lines.

The ſame demonſtrated by Dr. Small.

Let there be any number, *m*, of right lines AB, AC, AD,

AD, &c. (fig. 43, plate 2.) interſecting in a point A, ſo as to make all the angles round it equal; and from any point E, let perpendiculars EB, EC, ED, &c. be drawn to AB, AC, AD, &c. and let AE be joined, $2\,(EB^2+BC^2+ED^2$, &c.$)=m\cdot EA^2$.

This propoſition follows directly from the firſt caſe of Theor. 2. Let $m=3$. The points B, C, D are in a circle of which EA is the diameter, and therefore (Lem. II.) the arches BC, CD, DB, are equal. Therefore

$$2\,(EB^2+EC^2+ED^2)=4\cdot 3\cdot R^2=3\cdot EA^2\,\dagger.$$

† R is the radius of the circle ABC.

Cor. I. Demonſtrated by Dr. Small.

If AB, AC, AD, interſect one another in a given point A, and make all the angles round it equal; and if from any point E there be drawn perpendiculars to AB, AC, AD; and if the ſum of the ſquares of the perpendiculars be equal to a given ſpace, the point E will be in the circumference of a given circle.

The double of the given ſpace is $m\cdot AE^2$, therefore AE is given in magnitude, and ſince the point A is given, the point E is in the circumference of a given circle.

Cor. II. Added by Dr. Small.

If the circumference of a circle FGH, of which the radius is R, be divided into m number of equal parts, by the ſemi-diameters AF, AG, AH, &c. making with any diameter EN the angles FAE, GAN, HAE, &c. twice the ſum of the ſquares of the ſines, or coſines of theſe angles will be $=mR^2$.

Let m be $=3$.

$FK=EB$; $GL=EC$; $HM=ED$. Therefore $2\,(FK^2+GL^2+HM^2)=3\,EA^2=3R^2$.

In the ſame manner, $AK=AB$; $AL=AC$; $AM=AD$. Therefore $2\,(AK^2+AL^2+AM^2)=3EA^2=3R^2$.

ARTICLE

ARTICLE XXXV.

Demonstra ons to Lawson's *Propositions proposed in* ARTICLE XX.

PROP. XI.

Peletarius.

ANALYS . 137, *Plate* 10.

LET DG uch the circle in G; let H f the circle, and join HG, HD.

By hyp. $=ACB+CD^2$:

but $EDF=DG^2$:

therefore $DG^2=ACB+CD^2$,

or $DG^2+HG^2=ACB+CD^2+HA^2$;

but $DG^2+HG^2=HD^2$,

and $ACB+HA^2=HC^2$;

therefore $HD^2=HC^2+DC^2$;

and therefore the angle DCH is a right angle.

Q. Q. V.

SYNTHESIS.

Since the angle DCH is a right angle,

$HD^2=HC^2+DC^2$:

but $HD^2=GD^2+GH^2$;

therefore $GD^2+GH=DC^2+HC^2$,

take the square of GH or AH from each,

and $GD^2=EDF=ACB+CD^2$.

Q. E. D.

The

The same by Mr. Lowry.

Draw DAK and join BK.
By sim. Δ's $AC : AD :: AK : AB$:
therefore $BAC = DAK$:
but Eu. II. 3. $BAC + AC^2 = ACB$,
and $DAK + AD^2 = DAK + CA^2 + CD^2 = ADK$;
therefore $ADK = ACB + CD^2$:
but Eu. III, 36. Cor. $ADK = EDF$;
therefore $EDF = ACB^2 + CD^2$.

Q. E. D.

Mr. *Burdon's* demonstration is exactly the same as Mr. *Lowry's*.

The same by Mr. Swale.

ANALYSIS.

Join DB and let it meet the circle in I.
By hyp. $EDF = ACB + CD^2$,
but $EDF = IDB$;
therefore $IDB = ACB + CD^2$:
but $CD^2 = BD^2 - CB^2$;
wherefore $IDB = ACB + DB^2 - CB^2$:
but $DB^2 = IBD + IDB$,
and $CB^2 = ACB + ABC$;
therefore $IBD + IDB = ABC + IDB$;
wherefore $IBD = ABC$.

Now this we shall find to be true, if a circle be conceived to be drawn through the points A, C, D and I.

The same by Mr. Campbell.

Let BD be drawn intersecting the circle in I, and join IA.
By sim. Δ's $CB : DB :: IB : AB$;
therefore $ABC = DBI$,
or $CB \cdot (CB - CA) = DB \cdot (BD - DI)$,
or

or $CB^2 - ACB = DB^2 - BDI$.
that is, $IDB = ACB + DB^2 - CB^2$:
but $IDB = EDF$ and $DB^2 - CB^2 = CD^2$;
therefore $EDF = ACB + CD^2$.

Q. E. D.

PROP. XII.

Demonstrated by Peletarius.

ANALYSIS. Fig. 138, *Plate* 10.

By hyp. $2FHG = HD^2 + HE^2$:
but $2CH^2 + 2CD^2 = HD^2 + HE^2$;
therefore $FHG = CD^2 + CH^2$:
but, Prop. XI. $FHG = ACB + CH^2$,
therefore $CD^2 + CH^2 = ACB + CH^2$,
and therefore $CD^2 = ACB$.

Q. Q. V.

SYNTHESIS.

Since $ACB = CD^2$
$ACB + CH^2 = CD^2 + CH^2$:
but, Prop. XI. $ACB + CH^2 = FHG$;
therefore $FHG = CD^2 + CH^2$,
and $2FHG = 2CD^2 + 2CH^2$:
but $HD^2 + HE^2 = 2CD^2 + 2CH^2$;
therefore $2FHG = HD^2 + HE^2$.

Q. E. D.

The same by Mr. Lowry.

By Prop. XI. $FHG = ACB + CH^2$:
but, by hyp. $CD^2 = ACB$;
therefore $FHG = CD^2 + CH^2$,
and $2FHG = 2DC^2 + 2CH^2$:
but $HD = CD - CH$ and $HE = CD + CH$,
therefore $HD^2 + HE^2 = 2CD^2 + 2CH^2$;
and therefore $2FHG = HD^2 + HE^2$.

Q. E. D.

The

The same demonstrated by Mr. Burdon.

By Prop. XI. $CH^2 = FHG - ACB$,
and Eu. II. 5. $CH^2 = CE^2 - DHE$;
therefore $FHG - ACB = CE^2 - DHE$:
but, by hyp. $ACB = CD^2 = CE^2$;
therefore $2FHG = 4CE^2 - 2DHE$,
but Eu. II. 4. $DH^2 + HE^2 = 4CE^2 (DE^2) - 2HDE$;
consequently, $2FHG = DH^2 + EH^2$.

Q. E. D.

The same by Mr. Swale.

ANALYSIS.

Join DB, HB, and let HB meet the circle in I.
By hyp. $2FHG = HD^2 + HE^2$,
but $2FHG = 2IHB$;
therefore $2IHB = HD^2 + HE^2$:
but $HD^2 + HE^2 = DE^2 - 2DHE$
$= 4CD^2 - 2DHE = 4ACB - 2DHE$,
and $2DHE = 2DH \cdot (DH + 2CH)$
$= 2DH^2 + 4DHC = 2DB^2 - 2HB^2$
$= 2ACB - 2CH^2$.
Hence $2IHB = 2ACB + 2CH^2$,
and $IHB = ACB + CH^2$;
and therefore $FHG = ACB + CH^2$,
which is true by the eleventh Proposition.

Q. E. D.

The same by Mr. Campbell.

By Prop. XI. $FHG = ACB + CH^2$
$= CE^2 + CH^2 = HE^2 - 2HCE$
$= HE^2 - (HE + HD) \cdot \frac{1}{2}(HE - HD) = HE^2 - \frac{1}{2}(HE^2 - HD^2)$
$= \frac{1}{2}HE^2 + \frac{1}{2}HD^2$;
and therefore $2FHG = HE^2 + HD^2$.

Q. E. D.

PROP. XIII.

Demonstrated by Peletarius.

ANALYSIS. Fig. 139, 140. *Plate X.*

By hyp.	$DG^2 = EGF$:
but	$DG^2 = DC^2 + GC^2$,
and Prop. XI.	$EGF = ACB + GC^2$;
therefore	$DC^2 + GC^2 = ACB + GC^2$,
and therefore	$DC^2 = ACB$.

Q. Q. V.

SYNTHESIS.

Since	$DC^2 = ACB$,
therefore	$DC^2 + GC^2 = ACB + GC^2$:
but	$DC^2 + GC^2 = DG^2$,
and Prop. XI.	$ACB + GC^2 = EGF$;
therefore	$DG^2 = EGF$.

Q. E. D.

Conversely.

ANALYSIS.

By hyp.	$DC^2 = ACB$;
therefore	$DC^2 + GC^2 = ACB + GC^2$:
but Prop. XI.	$EGF = ACB + GC^2$,
and	$GD^2 = DC^2 + GC^2$;
therefore	$GD^2 = EGF$.

Q. Q. V.

SYNTHESIS.

By Analysis	$GD^2 = EGF$;
but	$GD^2 = DC^2 + GC^2$,
and Prop. XI.	$EGF = ACB + GC^2$,
therefore	$DC^2 + GC^2 = ACB + GC^2$,
and therefore	$DC^2 = ACB$.

Q. E. D.

The

The same by Mr. Lowry.

Prop. XI.	$EGF = ACB + CG^2$,
hyp.	$CD^2 = ACB$;
re	$EGF = CD^2 + CG^2 = DG^2$.

Q. E. D.

Conversely.

Prop. XI.	$EGF = ACB + CG^2$,
hyp.	$EGF = DG^2 = CD^2 + CG^2$;
re	$ACB = CD^2$.

Q. E. D.

The same by Mr. Burdon.

hyp.	$CD^2 = ACB$,
Prop. XI.	$CG^2 = EGF - ACB$;
re	$CD^2 + CG^2 = EGF$;
	$CD^2 + CG^2 = DG^2$;
re	$DG^2 = EGF$.

Q. E. D.

Conversely.

hyp.	$EGF = DG^2 = CD^2 + CG^2$,
Prop. XI.	$EGF = CG^2 + ACB$;
re	$CD^2 = ACB$.

Q. E. D.

The same by Mr. Swale.

ANALYSIS.

hyp.	$GD^2 = EGF$,
. I. 47.	$GD^2 = GC^2 + DC^2$;
re	$EGF = GC^2 + DC^2$:
p. XI.	$EGF = ACB + CG^2$;
re	$CD^2 = ACB$.

Q. Q. V.

Converſely.

By hyp. $CD^2 = ACB$,
and Eu. I. 47. $CD^2 = DG^2 - CG^2$;
therefore $ACB + CG^2 = DG^2$;
but Prop. XI. $ACB + CG^2 = EGF$;
wherefore $DG^2 = EGF$.

Q. Q. V.

The ſame by Mr. Campbell.

By hyp. $ACB + CG^2 = CD^2 + CG^2 = GD^2$,
and Prop. XI. $ACB + CG^2 = EGF^2$:
therefore $GD^2 = EGF$.

Q. E. D.

Converſely.

By hyp. $EGF = GD^2 = CD^2 + CG^2$,
and Prop. XI. $EGF = ACB + CG^2$;
therefore $CD^2 = ACB$.

Q. E. D.

PROP. XIV.

Demonſtrated by Peletarius.

ANALYSIS. Fig. 139, 140. *Plate X.*

Draw EL, FM parallel to CD and meeting CG in L, M.

By hyp. $gCH = CD^2$;
therefore $CH : CD :: CD : Cg$,
that is $FM : CD :: CD : EL$:
but $FM : CD :: GF : GD$,
and $CD : EL :: GD : GE$;
therefore $GF : GD :: GD : GE$;
wherefore $GD^2 = EGF$;
theref. conv. Prop. XIII. $CD^2 = ACB$.

Q. Q. V.

SYNTHE-

SYNTHESIS.

e	$CD^2 = ACB$;
p. XIII.	$GD^2 = EGF$;
re	$GF : GD :: GD : GE$:
	$FM : CD :: GF : GD$,
	$CD : EL :: GD : GE$;
re	$FM : CD :: CD : EL$,
	$CH : CD :: CD : Cg$;
refore	$gCH = CD^2$.

Q. E. D.

The same by Messrs. Lowry *and* Burdon.

. △'s	$DF : DG :: DH : DC$,
	$DE : DG :: Dg : DC$;
comp. et divid.	$GF : GD :: CH : CD$,
	$GE : GD :: Cg : CD$;
re	$EGF : GD^2 :: gCH : CD^2$:
rop. XIII.	$GD^2 = EGF$;
refore	$CD^2 = gCH$.

Q. E. D.

The same by Mr. Swale.

ANALYSIS.

hyp.	$gCH = CD^2$;
re	$gC : DC :: DC : HC$,
parallels	$gC : DC :: EG : DG$;
re	$DC : HC :: EG : DG$:
	$DC : DH : DG : DF$,
npounding	$DC : HC :: DG : GF$;
equality	$EG : DG :: DG : FG$;
re	$GD^2 = EGF$,

is true by the thirteenth proposition.

The same by Mr. Campbell.

u. VI. 2.	$GF : GD :: CH : CD$;
re	$GF^2 : GD^2 :: CH^2 : CD^2$.

Cc 3 But

But $GD^2 = CD^2 + CG^2 = ACB + CG^2$,
and Prop. XI. $ACB + CG^2 = EGF$;
therefore $GD^2 = EGF$.
Hence it will be $GF^2 : EGF :: CH^2 : CD^2$:
but $GF : GE :: CH : Cg$;
and therefore $GF^2 : EGF :: CH^2 : gCH$;
wherefore $CD^2 = gCH$.

Q. E. D.

PROP. XV.

Demonstrated by Peletarius.

ANALYSIS. Fig. 141, *Plate* 10.

Let F be the centre of the circle and join FD.
Since $AC : BC :: AE : BE$,
and AB is bisected in F;
by Prop. I. $CEF = AEB$;
add the square of EF to each
and $CFE = AF^2 = DF^2$;
therefore $CF : DF :: DF : EF$:
therefore the $\triangle$'s CDF, EDF, are equi-angular;
and therefore the $\angle$ CDF = the $\angle$ DEF;
but the $\angle$ DEF is a right-angle;
therefore the $\angle$ CDF is a right-angle,
and therefore CD touches the circle in D.

Q. Q. V.

SYNTHESIS.

Because CD touches the circle,
the $\angle$ CDF will be a right-angle:
but, the $\angle$ DEF is a right-angle;
therefore the $\triangle$'s CDF, DEF are equi-angular:
and therefore $CF : DF :: DF : EF$;
wherefore $CFE = DF^2 = AF^2$;
take the square of EF from each,
and $CEF = AEB$;
but AB is bisected in F;
theref. conv. Prop. I. $AC : BC :: AE : BE$.

Q. E. D.

Con-

Converſely.

ANALYSIS.

Let F be the centre of the circle and join FD.
Becauſe CD touches the circle,
the ∠ CDF will be a right-angle;
but, the ∠ DEF is a right-angle;
therefore the △'s CDF, DEF are equi-angular,
and therefore CF : DF :: DF : EF;
wherefore $CFE = DF^2 = AF^2$:
take the ſquare of EF from each,
and CEF = AEB;
but AB is biſected in F:
theref. conv. Prop. I. AC : BC :: AE : BE.

Q. Q. V.

SYNTHESIS.

Since AC : BC :: AE : BE,
and AB is biſected in F;
by Prop. I. CEF = AEB:
add the ſquare of EF to each,
and $CFE = AF^2 = DF^2$;
therefore CF : DF :: DF : EF:
therefore the △'s CDF, EDF are equi-angular;
and therefore the ∠CDF = the ∠DEF;
but the ∠DEF is a right-angle;
therefore the ∠CDF is a right-angle,
and therefore CD touches the circle in D.

Q. E. D.

Again Converſely.

ANALYSIS.

Let F be the centre of the circle, and join FD.
Becauſe DE is perpendicular to the diameter AB.
the ∠DEC will be a right-angle

but

but the $\angle$CDF is a right-angle;
therefore the $\triangle$'s CDF, DEF, are equi-angular;
therefore CF : CD :: CD : CE;
and therefore FCE $=$ CD^2 $=$ ACB;
wheref. conv. Prop. I. AC : BC :: AE : BE.

Q. Q. V.

SYNTHESIS.

By Analysis AC : BC :: AE : BE;
therefore Prop. I. FCE $=$ ACB $=$ CD^2;
therefore CF : CD :: CD : CE;
wherefore the $\triangle$'s CDF, DEF, are equi-angular;
therefore the $\angle$DEC $=$ the $\angle$CDF:
but the $\angle$CDF is a right-angle;
therefore the $\angle$DEC will be a right-angle;
wherefore DE is perpendicular to the diameter AB.

Q. E. D.

The same by Mr. Lowry.

Let F be the centre of the circle, and join FD.
By Eu. III. 18. the $\angle$CDF is a right-angle;
therefore the $\triangle$'s CDF, CDE are equi-angular;
and therefore FC or AC$+$AF : AF or BF :: DF or AF : EF;
theref. mixedly AC : BC :: AE : BE.

Q. E. D.

Conversely.

By hyp. AC : BC :: AE : BE,
that is, CF$-$AF : CF$+$AF :: AF$-$EF : AF$+$EF;
mixedly, CF : AF or DF :: DF : EF;
theref. Eu. VI. 6. the $\triangle$'s CDF, EDF, are equiangul.
therefore the $\angle$CDF $=$ $\angle$DEF $=$ a right-angle;
wheref. Eu. III. 16. Cor. CD touches the circle in D.

Q. E. D.

Again Conversely.

From what is done above the $\triangle$CDF, EDF are equi-angular.

But

But the $\angle CDF$ is a right-angle;
therefore the $\angle DEF$ will be a right-angle;
and therefore DE is perpendicular to AB.

Q. E. D.

The ſame by Mr. Burdon.

Biſect AB in F, and join DF;
Then by ſim. △'s $CF : CD :: CD : CE$;
therefore $ECF = CD^2 = ACB$;
wheref. conv. Prop. I. $AC : BC :: AE : BE$.

Q. E. D.

Converſely.

Becauſe $AC : BC :: AE : BE$;
by Prop. I. $ECF = ACB = CD^2$;
therefore $CF : CD :: CD : CE$;
wherefore Eu. VI. 6. $\angle CED = \angle FDC =$ a right-ang.

Q. E. D.

The ſame by Mr. Swale.

ANALYSIS.

Let F be the centre of the circle and join DF.
By hyp. $AC : BC :: AE : BE$;
theref. by diviſion $AC : BC - AC :: AE : BE - AE$,
that is $AC : 2AF :: AE : 2EF$,
or $AC : AF :: AE : EF$,
and by compoſition $CF : DF :: DF : EF$.

Q. Q. V.

Converſely.

ANALYSIS.

Since CD touches the circle in D;
by Eu. III. 36. $ACB = CD^2$;
therefore $CA : CD :: CD : CB$:

but

but Eu. VI. 8. CF : CD :: CD : CE;
therefore CF : CA :: CB : CE,
and by divifion CF : AF :: CB : EB.
Again, Eu. VI. 8. AE : DE :: DE : BE,
and CE : DE :: DE : FE;
hence, by divifi. &c. AC : AE :: AF : EF :: CF : AF:
but CF : AF :: CB : EB;
therefore AC : AE :: CB : EB,
or AC : BC :: AE : BE.

Q. Q. V.

Again Converfely.

ANALYSIS.

By Eu. VI. 8. CE : DE :: DE : FE,
and AE : DE :: DE : BE;
therefore CE : AE :: BE : FE,
and by divifion, &c. AC : AE :: AF : EF :: CF : AF:
but CF . AF :: BC : BE;
therefore AC : AE :: BC : BE,
or AC : BC :: AE : BE.

Q. Q. V.

The fame by Mr. Campbell.

From F the centre of the circle draw FD and join AD, BD.
By Eu. III. 18. FD is perpendicular to CD;
therefore the ∠CDF will be a right-angle.
but the ∠ADB is a right-angle;
wherefore the ∠CDF = the ∠ADB;
and therefore CEF = DE^2 = AEB;
wheref. conv. Prop. I. AC : BC :: AE . BE.

Q. E. D.

Converfely.

By hyp. AC : BC :: AE : BE;
therefore Prop. I. CEF = AEB = DE^2;

where-

wherefore $CE : DE :: DE : FE$;
therefore, the Δ's CDE, DEF are equi-angular,
and the $\angle CDE =$ the $\angle EFD$;
and theref. the $\angle CDF = CDE + EDF = EFD + EDF =$ a rt. ang.
wherefore, CD touches the circle in D.

Q. E. D.

Again Conversely.

By hyp. $AC : BC :: AE : BE$;
invertendo $BC : AC :: BE : AE$;
comp. et divi. $AC + BC : BC - AC :: BE + AE : BE - AE$,
that is $CF . DF :: DF : EF$;
therefore the Δ's CDE, DEF are equi-angular,
and therefore the $\angle CDE =$ the $\angle DEF$,
but the CDE is a right-angle;
therefore the $\angle DEF$ is a right-angle,
wherefore DE is perpendicular to AB.

Q. E. D.

PROP. XVI.

Demonstrated by Peletarius.

ANALYSIS. Fig. 142, *Plate* 10.

Join AD, BD,
By hyp. $AEF = DF^2$;
therefore $AE : DE :: DE : FE$;
and therefore the Δ's FED, AED are equi-angular;
wherefore the $\angle EDF = \angle EAD = \angle ECB$;
therefore DF is parallel to CB.

Q. Q. V.

SYNTHESIS.

Since DF is parallel to CB,
the $\angle EDF = \angle ECB = \angle EAD$;
therefore the Δ's FED, AED are equi-angular,
and

and therefore $AE : DE :: DE : FE$,
wherefore $AEF = DE^2$.

Q. E. D.

The same by Messrs. Burdon, Campbell, Lowry *and* Swale.

ANALYSIS, by Mr. Swale.

By hyp. $AEF = DE^2$;
therefore $FE : DE :: DE : AE$,
and by parallels $FE : DE :: BE : CE$;
therefore $DE : AE . : BE : CE$;
and therefore $CED = AEB$.

Q. Q. V.

SYNTHESIS, by Messrs. Burdon, Campbell *and* Lowry.

By Eu. III. 3. $AEB = CED$;
therefore $BE : CE :: DE : AE$,
and by sim. △'s $BE : CE :: FE : DE$;
therefore $DE : AE :: FE : DE$;
and therefore $AEF = DE^2$.

Q. E. D.

PROP. XVII.

Demonstrated by Peletarius.

ANALYSIS. Fig. 143, *Plate* 10.

Join BE, CE.
By hyp. $CFG : BF^2 :: CG : BD :: BCG : CBD$:
but, Prop. 16. $CFG = EF^2$ and $CBD = AB^2$;
therefore $EF^2 : BF^2 :: BCG : BA^2$;
and therefore $EF^2 : BCG :: BF^2 : BA^2 :: EF^2 : EC^2$;
wherefore $BCG = EC^2$;
therefore $BC : EC :: EC : GC$:
therefore the △'s BCE, CGE are equi-angular,
and

and therefore the $\angle CEB = $ the $\angle CGE$;
wherefore the $\angle CAB = \angle EGB = \angle ABC$
therefore CA is equal to CB.

Q. Q. V.

SYNTHESIS.

Since CA is equal to CB,
the $\angle CAB = \angle ABC = \angle BGE$;
therefore the $\angle CEB = $ the $\angle CGE$;
wherefore $BC : EC :: EC : GC$;
therefore $BCG = CE^2$;
and therefore $EF^2 : BCG :: EF^2 : EC^2 :: BF^2 : BA^2$;
but Prop. XVI. $EF^2 = CFG$ and $BA^2 = CBD$;
therefore $CFG : BCG :: BF^2 : CBD$,
or $CFG : BF^2 :: BCG : CBD :: CG : BD$.

Q. E. D.

The same by Messrs. Burdon *and* Lowry.

Join BE, CE, and produce GE to meet AC in I.
By sim. $\triangle$'s, $EF : BF :: EC : BA$,
that is $EF^2 : BF^2 :: EC^2 : BA^2$.
Again, by sim. $\triangle$'s, $AC : EC :: EC : IC$ or GC;
therefore $EC^2 = ACI = BCG$:
but, by hyp. & Prop. 16. $BA^2 = CBD$ & $EF^2 = CFG$.
Hence, $CFG : BF^2 :: BCG : CBD : CG : BD$.

Q. E. D.

The same by Mr. Swale.

ANALYSIS.

By hyp. $CFG : BF^2 :: CG : BD :: BCG : CBD$
but, Prop. 16. and hyp. $CFG = EF^2$ and $CBD = AB^2$,
and, Em. Geo. IV. 22. Cor. $BCG = CE^2$;
therefore $EF^2 : BF^2 :: CE^2 : AB^2$,
that is $EF : BF :: CE : AB$;
but by sim. $\triangle$'s $CE : AB :: CF : AF$;
therefore $EF : BF :: CF : AF$
and therefore $AFE = BFC$.

Q. Q. V.

The same by Mr. Campbell.

Produce FG to meet the circle in H and join CE, BE.

By Eu. I. 29.	the $\angle AEH =$ the $\angle EAB$;
theref. Eu. III. 26.	the arc $AH =$ the arc BE:
but, by hyp.	$AC = BC$;
therefore Eu. III. 28.	the arc $AC =$ the arc BC;
wherefore	the arc $CH =$ the arc CE;
therefore Eu. III. 17.	the $\angle CEH =$ the $\angle CBE$;
therefore	the $\triangle$'s CBE, CGE are equi-angul.
and therefore	$CG : CE :: CE : CB$;
therefore	$BCG = CE^2$.
Again, by sim. $\triangle$'s	$FE : CE :: CF : AB$,
that is	$FE^2 : CE^2 :: CF^2 : AB^2$:
but	$CE^2 = BCG$ and $AB^2 = CBD$,
and Prop. XVI.	$FE^2 = CFG$;
therefore	$CFG : BCG :: BF : CBD$;

permutando, $CFG : BF^2 :: BCG : CBD :: CG : BD$.

Q. E. D.

ARTICLE XXXVI.

A Problem, with its Investigation, by Mr. Colin Campbell, *of Kendal.*

PROBLEM, *Fig.* 144, *Plate* 10.

IF FGHI, KCML be two wheels revolving round the centres S, O, and connected by the flexible band FGHMLKF. It is required to determine the friction of that band on each wheel, supposing the centre S fixed, and the centre O urged by a force in the direction $SO = T$.

Postulate. If a given body slide over another given body with a given velocity, the friction arising from

from its motion, is as its weight or preſſure on the other body. (For the Demonſtration of this, ſee *Martin* and *Chamber's* Philoſophical Memoirs of the Royal Academy of Sciences at Paris for 1699, Eſſay 10.)

INVESTIGATION.

That the inveſtigation may be as eaſy as poſſible, let OA, OB, OC, &c. be a number of immovable radii over which the flexible line PABCDEP is ſtretched with the force T; draw ON, NQ perdicular to AB, BO, and the tenſion T in the direction BA is reſolved into BQ, QN, and the effect in BQ will be $T\times BN \div AO = T\times AB \div 2AO$.

Now, if the points A, B, C, &c. be infinitely near, and the force T communicated by the action of the band ariſing from the force acting on the centre O as per Problem, and the variable arc $KV = z$; then will the preſſure at $V = T\dot{z} \div 2OV$, and the fluent $= Tz \div 2OV$. Therefore $T\times\frac{1}{2}$ arc $KLM \div AO =$ the whole preſſure on the ſurface of the wheel KCML; and for the ſame reaſon $T\times\frac{1}{2}$ arc $FGH \div FS =$ the whole preſſure on the ſurface of the wheel FGHI; which *per Poſtulate* are as the effects of friction on each wheel.

Q. E. I.

Cor. I. The arc FGH being ſimilar to the arc KCM, it follows that the friction on the wheel FGHI is to that on the wheel KCML, as the arc FGH is to its ſupplement FIH.

Cor. II. If the band croſs itſelf between the two wheels, it is evident that the number of degrees to which it is applied on each wheel will be equal; and conſequently the friction on each will alſo be equal.

ARTICLE XXXVII.

Of finding the Sums of certain Series, by Mr. Stirling's *differential method, by Mr.* J. Mabbot, Manchefter.

(Continued from page 181.*)*

28. REquired the fum of 100 initial terms of the feries,

$2+5+8+11+14+\&c$?

Here $T=3z-1=-1+3z$, the values of z being 1, 2, 3, &c.

& $S=(-z+z+1)\cdot\frac{3}{2}z=\frac{3z^2+z}{2}=15050$, when $z=100$.

29. What is the fum of 25 terms of the feries,

$30+35+40+45+50+\&c$?

Here $T=25+5z$, the values of z being 1, 2, 3, &c.

and $S=25\dot{z}+(z+1)\cdot\frac{5}{2}z=\frac{5z^2+55z}{2}=2250$, when $z=25$.

30. Required the fum of z (10) terms of the feries

$(5+3)^2+(5+6)^2+(5+9)^2+\&c$?

Here $T=(3z+5)^2=9z^2+30z+25=25+39z+9z\cdot(z-1)$ the values of z being 1, 2, 3, &c. and $S=25z$ $+(z+1)\cdot\frac{39}{2}z+3z\cdot(z-1)=3z^3+\frac{39}{2}z^2+\frac{83}{2}z=5365$, when $z=10$.

31. Required the number of cannon-fhot in a fquare pile, the fide of which is 50 ?

The feries will be $1+2^2+3^2+4^2+5^2+\&c$.

Here $T=z^2=z+z\cdot(z-1)$ the values of z being 1, 2, 3, &c.

& $S=z+1\cdot(\frac{1}{2}z+\frac{1}{3}z\cdot z-1)=\frac{1}{6}z\cdot z+1\cdot 2z+1=42925$, when $z=50$.

32. Required the number of folid inches in a pyramid compofed of 1000 ftones of a cubical figure, the

the length of the side of the highest stone being one inch, of the second two inches, of the third three inches, &c.

The series will be $1+2^3+3^3+4^3+5^3+$&c.

Here $T=z^3=z+3z\cdot z-1+z\cdot z-1\cdot z-2$, the values of z being 1, 2, 3, &c.

and $S=\overline{z+1}\cdot(\frac{1}{2}z+z\cdot z-1+\frac{1}{4}z\cdot z-1\cdot z-2)=\frac{zz}{4}\times\overline{z+1}|^2$

$=250500250000$, when $z=1000$.

33. What is the sum of z (40) terms of the series $1\cdot2+3\cdot4+5\cdot6+7\cdot8+$&c ?

Here $T=\overline{2z-1}\cdot2z=4z^2-2z=2z+4z\cdot z-1$, the values of z being 1, 2, 3, &c.

and $S=\overline{z+1}\cdot(z+\frac{4}{3}z\cdot z-1=\frac{4z^3+3z^2-z}{3}=22920$, when $z=40$.

34. Required the sum of z (6) terms of the series $35\cdot85+40\cdot88+45\cdot91+50\cdot94+$&c ?

Here $T=\overline{5z+30}\cdot\overline{3z+82}=2460+515z+15z\cdot z-1$, the values of z being 1, 2, 3, &c. and $S=2460z+$

$\overline{z+1}\cdot(\frac{515}{2}z+5z\cdot z-1)=\frac{10z^3+515z^2+5425z}{2}=26625$ when $z=6$

35. What is the sum of z (10) terms of the series $15+40+77+126+$&c ?

The given series is the same as $3\cdot5+5\cdot8+7\cdot11+9\cdot14+$&c.

Here $T=(2z+1)\cdot(3z+2)=2+13z+6z\cdot z-1$, the values of z being 1, 2, 3, &c. and $S=$

$2z+\overline{z+1}\cdot(\frac{13}{2}z+2z\cdot z-1)=\frac{4z^3+13z^2+13z}{2}=2715$, when $z=10$

36. Required the sum of z terms of the series $1\cdot3\cdot5\cdot7\cdot9+3\cdot5\cdot7\cdot9\cdot11+5\cdot7\cdot9\cdot11\cdot13+$&c ?

Here $T=\overline{2z-1}\cdot\overline{2z+1}\cdot\overline{2z+3}\cdot\overline{2z+5}\cdot\overline{2z+7}$

$=32z^5+240z^4+560z^3+360z^2-142z-105$
$=-105+1050z+4200z\cdot z-1+2800z\cdot z-1\cdot z-2$
$+560z\cdot z-1\cdot z-2\cdot z-3+32z\cdot z-1\cdot z-2\cdot z-3\cdot z-4,$
the values of z being 1, 2, 3, &c.

and $S=-105z+z+1\times\dfrac{32z^5+352z^4+1288z^3+1592z^2-1142}{6}$

$=\dfrac{16}{3}z^6+64z^6+\dfrac{820}{3}z^4+480z^3+\dfrac{739}{3}z^3-124z.$

37. Find the sum of the infinite series

$$\frac{1}{3\cdot5}+\frac{1}{4\cdot6}+\frac{1}{5\cdot7}+\frac{1}{6\cdot8}+\&c.$$

Here $T=\dfrac{1}{z+2\cdot z+4}=\dfrac{1}{z\cdot z+1}-\dfrac{5}{z\cdot z+1\cdot z+2}$

$+\dfrac{12}{z\cdot z+1\cdot z+2\cdot z+3}-\dfrac{12}{z\cdot z+1\cdot z+2\cdot z+3\cdot z+4}$

the values of z being 1, 2, 3, &c.

and $S=\dfrac{1}{z}-\dfrac{5}{2z\cdot z+1}+\dfrac{4}{z\cdot z+1\cdot z+2}-\dfrac{3}{z\cdot z+1\cdot z+2+3}$

$=\dfrac{2z^2+7z+5}{2\cdot z+1\cdot z+2\cdot z+3}=\dfrac{7}{24}$, when z is taken $=1$.

38. What is the sum of the infinite series

$$\frac{6}{3\cdot5\cdot7}+\frac{6}{4\cdot6\cdot8}+\frac{6}{5\cdot7\cdot9}+\&c?$$

Here $T=\dfrac{6}{z+2\cdot z+4\cdot z+6}=\dfrac{6}{z\cdot z+1\cdot z+2}-\dfrac{54}{z\cdot z+1\cdots+2\cdot z+3}$

$+\dfrac{234}{z\cdot z+1\ldots\ldots z+4}-\dfrac{540}{z\cdot z+1\ldots\ldots z+5}+\dfrac{140}{z\cdot z+1\ldots\ldots z+6}$

the values of z being 1, 2, 3, &c.

and $S=\dfrac{3}{z\cdot z+1}-\dfrac{18}{z\cdot z+1\cdot z+2}+\dfrac{117}{2z\cdot z+1\ldots z+3}-\dfrac{108}{z\cdot z+1\ldots z+4}$

$+\dfrac{90}{z\cdot z+1\ldots z+5}=\dfrac{6z^3+48z^2+111z+69}{2\cdot z+1\cdot z+2\ldots z+5}=\dfrac{13}{80}$, when $z=1$.

ARTICLE

ARTICLE XXXVIII.

An easy method of constructing an azimuth by scale and compasses, by Mr. THOMAS KEITH, *author of a Treatise on Arithmetic, &c. &c.*

WITH the chord of 60° describe a circle, set off the complement of latitude from Z to P (fig. 145, pl. 10.) the complement of altitude from Z to *a* and draw *a* O *a*. Set off the polar distance from P to *p*, draw *pp*; from the intersection I draw IR parallel to ZN, and through the centre C draw CS parallel to *a* O, with C as a centre and distance O *a* cross IR in *m*, lastly draw Cn through *m*, then *n* S measured on a scale of chords will be the azimuth.

Note. In north latitude, *n* S is the azimuth from the south if IR fall on the left hand of ZN, but from the north if it fall on the right hand.

By this method an azimuth may at any time be constructed true to within half a degree, which is sufficiently exact for nautical purposes, and more simple than by either calculation or *Gunter*, vide *Robertson*'s Navigation, Art. 28th and 29th, Book IX. 5th edition. Required a demonstration of the above Article?

ARTICLE XXXIX.

Useful Propositions in Geometry,

By Mr. M. A. HARRISON.

PROP. I. THEO. *Fig.* 147, *Plate* 10.

IF the base AB of any plane triangle ABC be bisected by the diameter EF of its circumscribing circle, and from the point E a perpendicular be demited

mitted upon the longer side AC meeting it in P; then I say PC will be equal to half the sum and PA equal to half the difference of the sides of the triangle.

Dem. Join EA, EB, EC; make CG=CB, and join EG. Now the △s CBE, CGE, having the ∠ECB=∠ECG, CG=CB, and CE common to both, will also have EB (or EA) equal to EG; hence, because EP is perpendicular to AG, AP will be=PG; and therefore $AC+CB=AG+2CG=2PG+2CG=2PC$, and $AC-CB=AC-GC=AG=2AP$.

Q. E. D.

PROP. II. THEO.

If from C the vertex of any plane triangle ACB a line be drawn bisecting the vertical angle, and the perpendicular CD be demited upon the base; then I say the angle DCE, included between that line and the perpendicular, will be equal to half the difference of the angles at the base.

Dem. By Eu. III. 21, The $\angle CBA=\angle CEA$, & $\angle CAB=\angle CEB$; theref. $\angle CBA-\angle CAB=\angle CEA-\angle CEB=\angle BEL+\angle CEL-\angle CEB=2\angle CEL$.

But EL is parallel to CD and therefore $\angle CEL=\angle ECD$.

Q. E. D.

PROP. III. THEO.

If from the point P where a perpendicular from the extremity of the diameter of the circumscribing circle bisecting the base meets the longer side AC, a perpendicular PQ be demited upon the line bisecting the vertical angle: then I say that PQ will pass through L the middle of the base AB.

Dem. Because BC=CG and the ∠GCE=∠BCE;
CE will be perpendicular to BG;
and therefore PQ is parallel to BG
but AP=PG, therefore PQ bisects AB in L.

Q. E. D.

Cor. 1. If L, T (the point where CE cuts BG) be joined; then I say LT will be parallel and equal to PG.

Cor.

Cor. 2. I ſay, if BG, EP be produced they will meet the circumſcribing circle in the ſame point I.

Cor. 3. I ſay, that the ∠ABG (or AEI) is equal to the ∠DCE.

Cor. 4. The ∠AEG is = to the difference of the ∠'s at the baſe.

Cor. 5. I alſo ſay that the ∠EGB is = to the ∠ACD.
For, by Eu. III. 21. the ∠EIB=∠ECB,
and, by *Cor.* 3. of this Prop. the ∠IEG=∠DCK;
theref. ∠EGB (or ∠EIB∠+IEG)=∠ACD (or ∠ECB+∠DCK.)

PROP. IV.

If the perpendicular CD of any plane triangle ACB, be produced to meet the circumſcribing circle in R, and the perpendicular RS be demited upon the diameter EF; then I ſay, that the rectangle DLK is equal to the rectangle LES; and that the rectangle DLK is alſo equal to the ſquare of AP.

Demon. Draw CN parallel to RS and Join CF.
By Prop. III. Cor. 3, the ∠'s AEP, LEK, (NCF) are all equal,
and the ∠'s EPA, ELK, ENC are right ones;
therefore the △'s EPA, ELK, ENC are ſimilar,
and therefore NC or LD : AP :: EC : EA or EB,
and EC : EB or EA :: EB or EA : EK :: AP : LK;
wherefore NC or LD : AP :: AP : LK.
Again LD : NF or ES :: NE : LD :: LE : LK.
Hence LES=DLK=AP².

Q. E. D.

Cor. If upon LD a ſemicircle be deſcribed, and KV be drawn perpendicular to LD meeting the ſemi-circle in V, and LV be joined; then I ſay LV will be equal to AP, that is, equal to half the difference of the ſides of the triangle.

Note. K is the point where CE cuts the baſe AB.

(To be continued.)

ARTI-

ARTICLE XL.

Three Propofitions from *Lawfon*.

(To be anfwered in Number VI.*)*

PROP. XXII.

IF in AB the diameter of a circle be taken two points C and D fuch that AC : CB :: AD : DB, and D be without the circle, and DE perpendicular to AB, and through C be drawn any line meeting the circle in G and H, and the line DE in K, and GL touch the circle in G, and meet DE in L; then I fay the rectangle LDK is equal to the rectangle ADB.

PROP. XXIII.

If AB be the diameter of a circle and CD perpendicular thereto meeting it in C, and from the points A and B be inflected AE, BE to any point E in the circumference, meeting CD in F and G; I fay the rectangle GCF is equal to the rectangle ACB.

PROP. XXIV.

In AB the diameter of a circle let two points C and D be taken fuch that AC : CB :: AD : DB, and the point D be within the circle, and DE be perpendicular to AB, meeting the circumferenee in E and F, and let through C any line be drawn meeting the fame in G and H, and from the points G and H let GN, HN be inflected to any point in the fame N, and let them meet DE in M and L; I fay the rectangle LDM is equal to the fquare of DE.

ARTI-

ARTICLE XLI.

PROP. XXI. THEO. XVIII.

LET there be any number of right lines given by poſition, and let *a*, *b*, *c*, &c. be given magnitudes as many in number as there are right lines given by poſition, three right lines may be found that will be given by poſition, ſuch, that if from any point there be drawn perpendiculars to all the right lines given by poſition, and likewiſe there be drawn perpendiculars to the three lines found, the ſquare of the perpendicular drawn to one of the lines given by poſition, together with the ſpace to which the ſquare of the perpendicular drawn to another of the lines given by poſition has the ſame ratio that *a* has to *b*, together with the ſpace to which the ſquare of the perpendicular drawn to another of the lines given by poſition has the ſame ratio that *a* has to *c*, and ſo on, will be equal to the ſpace to which the ſum of the ſquares of the perpendiculars drawn to the three lines found has the ſame ratio that thrice *a* has to the ſum of *a*, *b*, *c*, &c.

PROP. XXII. THEO. XIX.

Let there be any regular figure of a greater number of ſides than three circumſcribed about a circle, and from any point in the circumference of the circle let there be drawn perpendiculars to the ſides of the figure; twice the ſum of the cubes of the perpendiculars, will be equal to five times the multiple of the cube of the ſemi-diameter of the circle by the number of the ſides of the figure.

ARTICLE XLII.

Answers to the Mathematical Questions proposed in ARTICLE XVII. No. II.

I. QUESTION 29, *answered by Mr.* S. Thornoby.

BY the rules for compound intereſt, the logarithm of the amount of £.1 for one year,
that is, the logarithm of £1·05 =·0211893
multiplied by the time 1796
the product is 38·0559828
to which add the logarithm of 1 penny, or the 1-240th. part of a pound } =−2·3802112
the ſum is the logarithm of the amount =35·6757710.

Now the index of this logarithm being 35, ſhews the number of figures, of which the amount of one penny in the given time doth conſiſt, to be 36, of which let it be ſufficient to expreſs the ſix firſt in figures, and the reſt in cyphers; then will the ſaid amount be

£·437992000000000000000000000000000000.

Now the value of a ſolid body, perfectly ſpherical, whoſe diameter is 8000 engliſh miles, (which is ſomewhat more than the diameter of the globe of our earth,) I ſay ſuch a ſolid body of fine gold would be in value about

£·2386600000000000000000000000.

From each of theſe great numbers let 23 cyphers be cut off; the remaining figures will be 4379920000000 in the amount of the penny; and 23866 in the value of the globe of gold. But 4379920000000 divided by 23866 is=183521327.

Hence it appears that one penny put out to uſe in the manner aforeſaid, would amount to more in value than one hundred and eighty-three millions, five hundred and twenty-one thouſand globes of fine ſolid gold, each bigger than the globe of the earth! a ſtrange and ſurpriſing, but no leſs certain truth! and this immenſe amount would be greatly increaſed by enlarging the rate of intereſt.

II. QUESTION 30, *anſwered by Mr.* J. H. Swale.

Let A=150=A's ſtock, a=A's time=14 months; B=B's ſtock unknown, b=B's time=12 months; C and c=C's ſtock and time, both unknown; m=195=A's, n=153=B's, p=127=C's, ſtock and gain reſpectively; w=475=aggregate of ſtock and gain, and w–A–B–C=the whole gain.

Now by the nature of *fellowſhip with time* we have

$$aA+bB+cC : w-A-B-C :: \begin{cases} aA : m-A \\ bB : n-B \\ cC : p-C\,; \end{cases}$$

and by multiplying means and extremes we have the following three equations

$$(aA+bB+cC)\cdot(m-A)=(w-A-B-C)\cdot aA,$$
$$(aA+bB+cC)\cdot(n-B)=(w-A-B-C)\cdot bB,$$
$$(aA+bB+cC)\cdot(p-C)=(w-A-B-C)\cdot cC.$$

From the firſt $(aA+bB+cC)\div(w-A-B-C)=aA\div(m-A)$,
and from the 2nd. $(aA+bB+cC)\div(w-A-B-C)=bB\div(n-B)$;
therefore $(n-B)\cdot aA=(m-A)\cdot bB$,
and $B=aAn\div(aA+(m-A)\cdot b)=121\cdot7045$.

Again from the firſt equation we have $cC=((w-A-B-C).aA-(aA+bB)\cdot(m-A))\div(m-A)$, and from the 3d. $cC=(aA+bB)\cdot(p-C)\div(w-A-B-p)$; hence by equating theſe two values of cC we ſhall obtain C=100, and conſequently $c=(aA+bB)\cdot(p-C)\div(w-A-B-p)\cdot C=12\cdot6$. Conſequently B's ſtock=£.121·7045, C's ſtock=£.100, and C's time =12·6 months. *W. W. R.*

And in like manner is the anſwer given by the Rev. Mr. L. Evans; *other anſwers were received, but they were not right!*

III. QUESTION 31, *anſwered by Mr.* J. Harris.

Let a=£.12. 10s. r=the amount of £.1 for half a year at the given rate per cent, T=54 half years, t=22 half years, and x=the half yearly purchaſe money required. Now it is evident that the amount

E e of

of x pounds for 11 years payable half-yearly, muſt be equal to the preſent worth of an annuity of £.25 per annum payable alſo half-yearly for 27 years.

Therefore $(a-a\div r^{T})\div(r-1)=(r^{t}x-x)\div(r-1)$; hence $x=(a-a\div r^{T})\div(r^{t}-1)=$ £13..19s..1d. the half-yearly purchaſe money. *W. W. R.*

And thus the anſwer is given by Meſſrs. Bulmer, Evans, Swale *and* Thornoby.

IV. QUESTION 32, *anſwered by Mr.* R. Simpſon.

To the given equation add $4r^4-r^2b^2$; and the ſquare root of the ſum gives

$$y^2+dy-2r^2=\pm r\surd(4r^2-b^2);$$

hence $$y=-\tfrac{1}{2}d\pm\surd(2r^2+\tfrac{1}{4}d^2\pm\surd(4r^2-b^2)).$$

The ſame anſwered by Mr. T. Bulmer.

The given equation being compared with the general one on page 155, *Simpſon's Algebra*, we find $a=2d$, $b=d^2-4r^2$, $c=-4r^2d$ and $d=r^2b^2$; and therefore $f=b-\tfrac{1}{4}a^2=-4r^2$ and $\tfrac{1}{2}af=-4r^2d=c$, which agrees with Mr. *Simpſon's* ſecond caſe;

hence $$y=-\tfrac{1}{4}a\pm\surd(\overline{\tfrac{1}{4}a}|^2-\tfrac{1}{2}f\pm\surd(\tfrac{1}{4}f^2-d))$$
$$=-\tfrac{1}{2}d\pm\surd(\overline{\tfrac{1}{2}d}|^2+2r^2\pm\surd(4r^2-b))$$

Meſſrs. Evans, Harris, Lowry, Surtees, Swale *and* Thornoby, *ſent anſwers to this queſtion.*

V. QUESTION 33, *anſwered by* Apollonius Junior.

Analyſis. Suppoſe the thing done and that ABC (fig. 148, pl. 10.) is really the triangle required, AC the given baſe, and CAB the given angle. On AC drop the perpendicular BP and produce it 'till PD be equal to the given difference; draw DE parallel to AC and let it meet BA produced in E; make the angle BEF equal to the angle BED; then the perpendicular BF being demitted, it will be equal

to

to BD, that is, equal to BC; therefore B is the centre of a circle passing through the given point C and touching the right lines ED, EF given by position in the points D and E. Consequently B (the vertex of the triangle) may be found by PROP. VIII. *Lawson's Translation of Apollonius on Tangencies.*

The composition by Mr. J. H. Swale.

Conf. At any point N on the indefinite right line LK erect the perpendicular NA = the given difference; draw AE meeting NL in E making the ∠NEA = the given one, and draw AC parallel to NK and = the given base; join EC, to which, from A, apply AO = AN, then drawing CB parallel to OA meeting EA produced in B; I say ABC will be the triangle that was to be constructed.

Demon. It is evident that BAC is = to the given angle, and AC = to the given base.

Now demit the perpendicular DB.

Then by sim. Δ's EA : EB :: AN : BD,

and EA : EB :: AO : BC;

therefore AN : BD :: AO : BC:

but AN = AO, and therefore BC = BD;

wherefore BC-BP=PD=AN=the given difference.

Q. E. D.

Otherwise by Mr. A. Buchanan.

Conf. Draw AC = the given base, and make the angle CAB = the given one; with the centre C and radius = the given difference of BC, BP describe a circle, and draw AM making the ∠BAM=∠BAC; then by *Prob.* XLIV. on page 249, *Simpson's Geometry*, find B the centre of a circle to touch AC, AM and the circle about the centre C; join BC and ABC will be the triangle required.

According to one or other of these methods nearly, is the Problem constructed by Messrs. Burdon, Elliott, Lowry, Simpson, Surtees *and* Thornaby.

VI. QUESTION 34, *anſwered by Mr.* Elliott.

Let ADBC (fig. 149, pl. 10.) repreſent a ſection of half the caſk. Put $AC = x$, then BD will be $= \frac{3}{4}x$ and $AE = \frac{1}{2}(AC + DB) = \frac{7}{8}x$.

But $AE = (BE+8)^2 - BE^2 = 16BE + 64 = \overline{\tfrac{7}{8}x}|^2$;

therefore $BE = \left(\frac{49}{64}x^2 - 64\right) \div 16 = \frac{49}{64 \times 16}x^2 - 4$.

Hence by Mr. *Lowry's* general rule Art. IV.

we have $\left(\frac{98}{64 \times 16}x^4 - 8x^2\right) \times \cdot 0023557 = 98$

or $$x^4 - \frac{64^2}{49}x^2 = \frac{64 \times 16}{\cdot 0023557}.$$

Whence $x = AC = 26 \cdot 5$ inches the bung diameter,
$BD = 19 \cdot 9$ inches the head diameter,
and $2AE = 59 \cdot 2$ inches the length of the caſk.

Meſſrs. Lowry, Simpſon, Surtees, Swale, Thornoby *and* Wood, *favoured us with ingenious ſolutions to this queſtion.*

VII. QUESTION 35, *anſwered by the Rev. Mr.* L. Evans.

Let ABCD (fig. 150, pl. 10.) be the ſpherical ſquare whoſe ſide is given; draw the arches AC, BD, which will interſect each other at right angles at O. Upon one of the ſides demit the perpendicular arch OE. Then in the right-angled ſpherical triangle OCE there is given the $\angle COE = 45°$ and EC = half the given ſide $= 14° \ 8'$ to find OE the radius of the inſcribed circle $= 14° \ 35'$ or $165° \ 25'$ and the $\angle OCF = 46° \ 49'$ or $133° \ 11'$. Hence the ſum of the angles of the ſquare is $= 374° \ 32'$, or $1065° \ 25'$;

con-

consequently $pr^2 \times 14^\circ 32' \div 180^\circ$ } =the area of the squa.
or, $pr^2 \times 705^\circ\ 28' \div 180^\circ$ }

and their sum $= 4pr^2 =$ the whole surface of the sphere; where $p = 3.1416$ and r the radius of the sphere.

In nearly the same manner is the answer given by Messrs, Elliott, Lowry, Simpson, Swale *and* Thornoby.

VIII. QUESTION 36, *answered by Mr.* Lowry.

Let A, B, C, (fig. 89, pl. 7.) be the three given places and D the fourth place which is required; then since the latitudes and longitudes of the places A, B, C are given, the distances AB, BC, AC and the angles BAC, ABC, ACB are easily found by trigonometry.

Let the arches AD, BD, CD be drawn, and by PROP. XXVIII. Art. 24, the sum of these arches will be the least possible when the angles ADB, BDC, ADC are equal to each other, each being equal to 120°. Now the position of the point D may be determined either by the intersection of two ellipses, projected as is taught in the prize question *Gent. Diary*, 1795; or by *Prob.* 152, Book II. of *Emerson's Algebra*; or it may be determined otherwise thus. Put s and c for the sine and cosine of the $\angle$BAC, m and $n =$ the sine and cosine of 120°, $d =$ sine of AB, $f =$ sine of AC, $q =$ cosine of BC, and x and $y =$ the sine and cosine of the $\angle$BAD.

Then by trigono. $m : d :: x : dx \div m =$ the sine of BD, and therefore its cosine is $= \sqrt{(1 - d^2x^2 \div m^2)}$.

Again $sy - cx =$ the sine of the $\angle$CAD, and theref. $m : f .: sy - cx : (sy - cx) f \div m =$ the sine of CD, and theref. its cosine will be $= \sqrt{(1 - \overline{sy - cx}|^2 \times f^2 \div m^2)}$.

 Hence

Hence, fine BD $\times$ fine CD $\times n +$ cos. BD $\times$ cos. CD $=$ cos. CB, *i.e.*

$(syx-cx^2)\cdot(df\div m^2)\cdot n+\sqrt{(1-d^2x^2\div m^2)}\times\sqrt{1-\overline{sy-cx}^2\times f^2\div m^2)}=q$

From which equation, by writing $\sqrt{1-x^2}$ for y, the value of x may be found and confequently the latitude and longitude of the place D.

The anfwers from Meffrs. Elliott, Simpfon, Swale, *and* Thornoby, *are very little different from the above.*

IX. QUESTION 37, *anfwered by Mr.* Lowry.

Conf. Let ABC (fig. 151, pl. 10.) be the triangle; from the points D and Q, where perpendiculars from the angular points meet the oppofite fides, demit the perpendiculars DI, QH upon the bafe; and divide the bafe AC in P fo that AP : PC :: DI : QH; I fay P is the point required.

Demon. Draw the perpendiculars PE, PF; then fince the triangle ABC is of a conftant magnitude, and the trapezium PEBFP a maximum, the fum of the triangles APE, CPF muft be a minimum.

To prove which, take any other point R in the bafe and draw the perpendiculars RK, RL.

Then by fim. $\triangle$'s $AC^2 : \triangle APC :: AP^2 : \triangle AEP$,

or $AC : \frac{1}{2}QH :: AP^2 : \triangle AEP = AP^2 \times QH \div 2AC$,

and $AC : \frac{1}{2}DI :: CP^2 : \triangle CPF = CP^2 \times DI \div 2AC$;

therefore the fum of the triangles AEP, CPF will be $= AP^2 \times QH \div 2AC + CP^2 \times DI \div 2AC$; and in like manner the fum of the $\triangle$'s AKR, CLR will be $= AR^2 \times QH \div 2AC + CR^2 \times DI \div 2AC$.

But *Prop.* C. Art. 34, $AP^2 \times QH \div 2AC + CP^2 \times DI \div 2AC + \frac{1}{2}PR^2$ is $=$ to $AR^2 \times QH \div 2AC + CR^2 \times DI \div 2AC$; therefore the fum of the triangles APE, CPF is lefs than the fum of the triangles AKR, CLR, by half the fquare on PR; and as this happens wherever the point R is taken, it follows that P is the point required.

Q. E. D.

A flux-

A fluxionary solution by Mr. Swale.

Let ABC be the given $\triangle$, P the required point, and PEBFP the trapezium. Demit the perpendicular Bd; put AC$=a$, B$d=d$, AP$=x$, let m and c be the sine and cosine of the $\angle$BAC, and n and s the sine and cosine of the $\angle$BCA; then PC$=a-x$, PE$=mx$, AE$=cx$, PF$=(a-x)\cdot n$, CF$=(a-x)\cdot s$, and the area of the triangle $=\frac{1}{2}ad$.

Again, the area of the$\triangle$AEP$=\frac{1}{2}cmx^2$.

and the area of the$\triangle$CPF$=\frac{1}{2}ns\cdot(a-x)^2$.

Hence the area of the trapezium PEBFP$=\triangle$ ABC$-\triangle$AEP$-\triangle$CPF$=\frac{1}{2}ad-(cm-ns)\cdot\frac{1}{2}x^2+ansx-\frac{1}{2}a^2ns=$a maximum, per quest.

This put into fluxions and reduced gives $x=ans \div (cm+ns)$.

Cor. When the $\angle$ ABC becomes a right-angle, x will be$=\frac{1}{2}a$.

And in a manner equally ingenious is the answer given by Messrs. Elliott, Simpson, *and* Thornoby.

X. QUESTION 38, *answered by Messrs.* Elliott *and* Lowry.

Case 1. When the sum of the squares is given (fig. 152, pl. 10.).

Conf. Let AB, CD be the parallel lines given by position and Pp the given curve; from any point E in CD draw EF, EK to make the given angles with CD, AB; draw EL perpendicular to EF and equal to EK, join FL, and with a distance equal to the side of the given square and centre E, describe a circle cutting FL in H; draw HI perpendicular to EF; then IP being drawn parallel to AB will meet the curve in the required point P.

Demon. Draw IG and PR parallel to EK, and PQ parallel to IE and join EH.

Then $\angle$PQC$=\angle$IEC and $\angle$PRA$=\angle$EKA$=$the given ones.

And

And by sim. Δ's $FI : IH :: FE : EL$,
and $FI : IG :: FE : EK$;
therefore $IH : IG :: EL : EK$;
but $EL = EK$; therefore $IH = IG$,
and by parallels $PR = IG$ (or IH) and $PQ = IE$;
therefore $PQ^2 + PR^2 = IE^2 + IH^2 = EH^2 =$ the given square by construction.

Case 2. When the difference of the squares is given (fig. 153, plate 10.).

Const. From any point E in AB draw EF, EH to make the given angles with AB, CD; draw EL perpendicular to EF, on which take EO equal to the side of the given square; then with the centre E and distance EH describe a circle intersecting the right line joining the points O, F in N; join EN and parallel thereto draw OI meeting EF in I, then IP being drawn parallel to AB will meet the curve in the required point P.

Demon. Draw IG and PQ parallel to EH and PR parallel to EF.

Then $\angle PQC = \angle FHC$ and $\angle PRA = \angle FEA =$ the given ones.
And by sim. Δ's $FI : IO :: FE : EN$,
and $FI : IG :: FE : EH$;
therefore $IO : IG :: EN : EH$;
but $EN = EH$, and therefore $IO = IG$;
and by parallels $PR = IE$ and $PQ = IG$;
therefore $PQ^2 - PR^2 = IG^2$ (or IO^2) $- IE^2 = EO^2 =$ the given square by Construction.

Q. E. D.

Mr. J. H. Swale *by a very ingenious Analysis reduces the Problem to very near the same as Prop.* 6. Simpson's Exercises, *Mr.* Thornoby *also answered it.*

XI. QUESTION 39, *answered by Mr.* Swale.

It is well known that the area of the figure of the versed sines in a semi-circle, whose radius is r, is =

$=3\cdot1416r^2=1413\cdot7168$ by the queſtion; hence $r=21\cdot2132$: whence the following compoſition.

With the radius now found deſcribe a circle, and draw any diameter LV (fig. 154, pl. 10.) on which produced take $VC=\frac{1}{2}VL$; from C draw the tangents CH, CI meeting a tangent at the point L in the points B, A; draw OE parallel to LB meeting the circle in E, and draw FE parallel to LV meeting AB in F; join FO and produce it to meet the circle in T and through T let a tangent be drawn to meet FA, FE produced in G, D: then ACB, GDF will be the Δ's whoſe difference is required.

Calculation. Since $LC=3LO=63\cdot6396$ is given, the ſide of the equilateral Δ is eaſily found to be $=73\cdot4846$, and its area $=\frac{1}{2}LC\times AB=2338\cdot26527508$.

Again, $GD=GF\sqrt{2}$, and it is well known that $2GF-GD$ (or $\overline{2-\sqrt{2}}\cdot GF)=LV$; hence $GF=LV\div(2-\sqrt{2})=72\cdot4247$, and therefore the area of the right-angled Δ $GDF=\frac{1}{2}GF^2=2622\cdot668585045$; whence the required difference is $284\cdot403309965$.

Meſſrs. Harris, Elliott, Gregory, Simpſon, Surtees, *and* Thornoby *alſo ſent anſwers.*

XII. QUESTION 40, *anſwered by Mr.* Lowry.

Conſ. On the indefinite right line CDR (fig. 155, pl. 10.) take CR equal the given ſum of the ſides, which biſect in D, and erect the perpendicular DI ſo that DI may be to DC as the given rectangle is to the area of the triangle; join IC and draw CBK making the $\angle ICK=\angle ICD$; on CD conſtitute the parallelogram DCKH = twice the given area; divide CR in A ſo that the rectangle CAR may be = the rectangle DCK, and make CB = AR; then join AB and ACB will be the triangle required.

Demon. About the ΔACB deſcribe a circle, and through the centre draw IEFG, which will biſect AB in E; draw CF parallel and Cp perpendicular to AB, and join AG, AI, CG, and DK.

Then

Then $AC+CB=AC+AR=CR=$ the given ſum of the ſides, and $AC \cdot CB=DC \cdot CK$; therefore the $\triangle ACB=\triangle DCK=$ half the parallelogram DCKH $=$ the given area.

Again $\quad AQ \cdot QB=AE^2-EQ^2=AE^2-AD^2$;
and by ſim. $\triangle$'s $\quad AD:CG::AI:GI::AE:AG$,
that is, $\quad AD:AE::CG:AG$,
or, $\quad AD^2:AE^2::CG^2:AG^2$,
or, $\quad AD^2:AE^2::FGI:EGI::FG:EG$;
theref. $\quad AE^2-AD^2:AE^2::EF:EG::AEF\,(\triangle ACB):AEG$,
or $\quad AE-AD^2:AEF::AE:EG::DI:DC$:
but $\quad DI:DC::$ the given rectang. $:\triangle ACB(AEF)$;
theref. $AQ \cdot QB=AE^2-AD^2=$ the given rectangle.

Elegant conſtructions to this Problem were given by Meſſrs. Swale *and* Thornoby.

XIII. QUESTION 41, *anſwered by Mr.* Lowry.

Analyſis. Conceive the Problem to be ſolved, and that ACB (fig. 147, pl. 10.) is really the triangle to be conſtructed; make CD perpendicular to AB, and produce it to meet the periphery of the circumſcribing circle in R and draw RS parallel to AB meeting the diameter ESLF perpendicular to the baſe, in S; join AR, RB.

Since $AC \cdot CB=FE \cdot CD$ and $AD \cdot DB=RD \cdot DC$, the ratio of FE to DR will be given: and becauſe the vertical angle ACB is given, the ratio of FE to AB will be the given, and therefore the ratio of AB to DR will be given: but the angle ARB($=$ the ſupplement of the angle ACB) is given, and therefore (*Sim. Ed. of Eu. Data, Prop.* LXXVIII.) the triangle ARB is given in ſpecies, and conſequently the triangle ACB is given in ſpecies.

A triangle ſimilar to the required one may be found thus; on any line AB having deſcribed a circle AFCB to contain the given vertical angle, through the centre draw FLE at right-angles to AB, and take LS to FE as the rectangle of the ſegments to the rectangle of the ſides; draw SR parallel and

CDR

CDR perpendicular to AB meeting the periphery in R and C; join AC, CB and ACB will be the triangle that was to be found, as is evident from the *analysis*.

The same answered by Mr. Buchanan.

Conf. Upon any line AB (fig. 150, pl. 10.) describe the segment of a circle capable of containing the given angle, and having completed the circle and drawn the diameter GG, apply DE perpendicular to AB such, that M : N :: GG : DE (M : N being the given ratio of the rectangles,) and produce it to cut the circle in C; join AC, CB and make CS = CB; draw BS, and produce CD to M, so that CD : CM :: AS : the given difference of the sides; then produce CA, CB to cut OQ parallel to AB, and OCQ will be the triangle required.

Demon. The vertical $\angle$ is evidently = the given one; now draw QR parallel to BS.

Then by sim. Δ's and *conf.* AS : OR :: AB : OQ :: CD : CM :: AS : the given difference of the sides; therefore OR (= OC — CQ) = that difference.

Again, by *conf.* M:N::GG:DE :: CD·GG:CDE::ACB:ADB;
but, by sim. Δ's AC : BC :: OC : QC,
and AD : BD :: OM : MQ;
therefore $AC^2 : OC^2 :: ACB : OCQ$,
and $AD^2 : OM^2 :: ADB : OMQ$;
therefore ACB : ADB :: OCQ : OMQ :: M : N.

Q. E. D.

The same answered by Mr. R. Elliott.

Conf. On any line as AB (fig. 150, pl. 10.) let a segment of a circle be described to contain the given angle; complete the circle and draw A*g* to the centre, and make *g*N perpendicular to AB; take NF a fourth proportional to $m \times AN$, $n \times Ag$ and

Then AC+CB=AC+AR=C... ...nd draw FE
sum of the sides, and AC·CB=D... ...e in E; make
the △ACB=△DCK=half the r... ...CB: lastly, take
= the given area. ... CB, CD, and the
Again AQ·QB= ... ...d through M draw
and by sim. △'s AD:C ... OCQ will be the tri-
that is, AD· ... ...ucted.
or, AD ... that the △ ACB is similar to
or, A ... ...ence it remains only to prove
theref. AE²–AD² ... AD·DB in the given ratio.
or AE–AD ...
but DI:D...AB(2AN):DE(FN)::m×AN:n×Ag,
theref. AQ· ... 2Ag×CD:DE×CD::m:n;

Elegant ... CD=AC×CB & DE×CD=AD×BD:
Messrs. S... AC×CB:AD×BD::m:n.

Q. E. D.

XII... *...ant constructions were also received from Messrs.* ...*ake and* Thornoby.

...IV. QUESTION 42, *answered by Mr.* Mabbott.

The equation defining the terms of the series is

$$T=\overline{-1}|^{x-9}\times\frac{1}{x\cdot x+2\cdot 2x-6\cdot 2x+10}=\overline{-1}|^{x-9}\times\frac{1}{2x\cdot x+2\cdot x-3\cdot 2x+10},$$

the values of the indeterminate quantity x being 9, 10, 11, &c.

But it is evident from the introduction to *Stirling's* Series, that this series cannot be summed from this differential equation; therefore put $z=x-3$ as being the least of the factors. Then

$$T=\overline{-1}|^{x-9}\times\frac{1}{2x\cdot x+2\cdot x-3\cdot 2x+10}=\overline{-1}|^{z-6}\times\frac{1}{z\cdot z+5\cdot 2z+6\cdot 2z+16}$$

comparing this with the general series (*vide* pa. 11, 12, *Stirling's Sum. Ser.*) and making the proper substitutions we shall find

$$T=\overline{-1}|^{z-6}\Big(\times\frac{1}{4z\cdot z+1\cdot z+2\cdot z+3}\;\frac{5}{2z\cdot z+1\ldots\ldots\cdot z+4}$$

+

$$+\frac{18}{z\cdot z+1\ldots\ldots\cdot z+5}-\frac{42}{z\cdot z+1\ldots\cdot z+6}+\frac{84}{z.z+1\ldots\ldots\cdot z+7}$$

$$-\frac{84}{z\cdot z+1\ldots\ldots\cdot z+8}\Big),$$ x being $=-1$ and the values of z 6, 7, 8, &c. Comparing again, this with the theorem in *Stirling's* 3rd. Prop. pa. 30, and making the neceſſary ſubſtitutions we ſhall at laſt find

$$S=\overline{-1}|^{z-6}\times\frac{z^4+14z^3+67z^2+126z+84}{8z\cdot z+1\ldots\ldots\ldots\ldots\cdot z+7}=\frac{631}{34594560}$$

(when $z=6$) the ſum required.

And thus the anſwer is given by Mr. Thornoby.

XV. QUESTION 43, *anſwered by Mr.* Ralph Simpſon, Sunderland Bridge.

Let $a=$ the earth's radius $=3985$ miles, $e=$ its denſity $=9$, $d=$ the denſity of the comet which is ſuppoſed to be $=$ the moon's $=11$, $s=$ the ſine of the comet's apparent ſemidiameter, as ſeen from the earth being $31'\ 14''$. Then $^*15\,ds^3a\div4e=$ 72·36 feet, will be the height of the tide required.

* *Vide Number* V. *when publiſhed.*

XVI. QUESTION 44, *anſwered by Mr.* Lowry.

Produce the tangents *ab*, *bc*, *dc*, *de* and *ef* 'till they meet in I, K, L, W, (fig. 146, pl. 10.) and let P and Q repreſent the principal axes of the ellipſis: join AE, *be*, BD, OI, OK, OW and OL. Then becauſe *af* is biſected in F it is parallel to AE. and ſince A$a=$Ab, and E$f=$Ee, *eb* will alſo be parallel to AE, and in the ſame way it may be ſhewn that *dc*, BD, *eb*; *bc*, AC, IL; *de*, EC, IK are parallel to each other reſpectively.

But IL, IK and LK are biſected in E, A and C, therefore $ab=\frac{1}{3}$IK, $cd=\frac{1}{3}$LK and $ef=\frac{1}{3}$IL.

More-

Moreover $2IM = 3IF$ or IC;
hence, $3 : 2 :: IM : IF :: AE : af = \frac{2}{3}AE = \frac{1}{3}KL$;
and in like manner, $bc = \frac{1}{3}IL$ and $de = \frac{1}{3}IK$;
therefore $ab^2 + bc^2 + cd^2 + de^2 + ef^2 + af^2 = \frac{2}{9}(IL^2 + IK^2 + LK^2)$.
But $IL^2 + IK^2 = 2LC^2 + 2IC^2$ and $LK^2 = 4LC^2$;
therefore $IL^2 + IK^2 + LK^2 = 6LC^2 + 2IC^2$.

Now by Emerſ. Conics I. 47. Cor. $LC^2 = \frac{3}{4}XY^2$ (XY being conjugate to CF) and $IC^2 = \frac{9}{4}CF^2$;
therefore $6LC^2 + 2IC^2 = \frac{9}{2}(XY^2 + CF^2)$,
hence, $ab^2 + bc^2 + cd^2 + de^2 + ef^2 + af^2 = XY^2 + CF^2$
$P^2 + Q^2$.

In the ſame way may be ſhewn that
$pq^2 + qr^2 + rs^2 + st^2 + tu^2 + ua^2 = P^2 + Q^2$.
theref. $ab^2 + bc^2 + cd^2 +$ &c. $= pq^2 + qr^2 + rs^2 +$ &c.

Again ſince AE, BD and BE are biſected in M, N, & O, AB & DE are each parallel and = to $MN = \frac{1}{2}FC$;
therefore $AB^2 + DE^2 = \frac{1}{2}FC^2$,
and $BC^2 + CD^2 = 2NB^2 + 2CN^2 = 2AM^2 + 2MF^2 = AF^2 + EF^2$;
theref. $AB^2 + BC^2 + CD^2 + DE^2 + EF^2 + AF^2 = \frac{1}{2}FC^2 + 4AM^2 + 4MF^2$, but $4MF^2 = \frac{1}{4}FC^2$ and $4MA^2 = LC^2 = \frac{3}{4}XY^2$;
theref. $AB^2 + BC^2 + CD^2 + DE^2 + EF^2 + AF^2 = \frac{3}{4}(XY^2 + CF^2) = \frac{3}{4}(P^2 + Q^2)$; in like manner $PQ^2 + QR^2 + RS^2 + ST^2 + TV^2 + VP^2 = \frac{3}{4}(P^2 + Q^2)$; theref. $AB^2 + BC^2 + CD^2 +$ &c. $= PQ^2 + QR^2 + RS^2 +$ &c.

Moreover $AO = OD$, $OB = OE$, and $OC = OF$,
theref. $OA^2 + OB^2 + OC^2 + OD^2 + OE^2 + OF^2 = 2OA^2 + 2OE^2 + 2OF^2 = 4AM^2 + 4OM^2 + 2OF^2 = \frac{3}{4}(XY^2 + CF^2) = \frac{3}{4}(P^2 + Q^2)$;
in like man. $OP^2 + OQ^2 + OR^2 + OS^2 + OT^2 + OV^2 = \frac{3}{4}(P^2 + Q^2)$;
theref. $OA^2 + OB^2 + OC^2 +$ &c. $= OP^2 + OQ^2 + OR^2 +$ &c.

Laſtly $Oa = Od$, $Ob = Oe$ and $Oc = Of$,
theref. $Oa^2 + Ob^2 + Oc^2 + Od^2 + Oe^2 + Of^2 = 2Oa^2 + 2Of^2 + 2Oe^2 = 4aF^2 + 4OF^2 + Oe^2 = CF^2 + 3OE^2 = CF^2 + XY^2 = P^2 + Q^2$:
in like manner $Op^2 + Oq^2 + Or^2 + Os^2 + Ot^2 + Ou^2 = P^2 + Q^2$;
theref. $Oa^2 + Ob^2 + Oc^2 +$ &c. $= Op^2 + Oq^2 + Or^2$ &c.

Q. E. D.

I have ſuppoſed the number of ſides of each polygon to be ſix; if any other number had been ſuppoſed the demonſtration would have been very little different; for, the ſum of the ſquares is always either equal

equal to, or has a conſtant ratio to the ſum of the ſquares of the axes of the ellipſis.

XVII. QUESTION 45, *anſwered by Mr.* Lowry.

An elegant demonſtration to this propoſition may be ſeen in *Dr. Stewart's* Tracts, page 50, from whence the whole of Pappus' compoſitions have been taken! The following demonſtration is ſomewhat different from Dr. Stewart's.

Draw the conjug. CS and join GE (fig. 157, pl. 10). Becau. DG(DM)+EG=AB=DM+MF, GE will be =to MF. Hence, by the circ. $DM^2 : MO^2 :: DM : MF :: DG : GE$ but, *Em. Conics* I. 24, $DK^2 : CS^2 :: DG : GE$;

therefore $DK : CS :: DM : MO :: DN : NQ$.

In like manner $DL : CS :: DN : NP$;

therefore $DK : DL :: PN : NQ$.

Q. E. D.

Cor. By Dr. *Stewart*, $DG^2 : ADB :: DG : GE$.

XVIII. QUESTION 46, *anſwered.*

Meſſrs. Elliott, Lowry, Sanderſon, Simpſon, Swale *and* Thornoby *agree in ſaying that this queſtion is not properly limited.*

XIX. QUESTION 47, *anſwered by Mr.* I. T. M'Donald, Durham.

Draw AKNH (fig. 158, pl. 10.) perpendicular and ML parallel to the baſe, put $BC=a$, $AH=d$, $NH=v$ and LM $(=(ad-av)\div d)=y$. Then by Emerſon's Fluxions, page 344, the fluent of $yv^2\dot{v}\div d$ or $(adv^2\dot{v}-av^3\dot{v})\div d^2=$(when $v=d$) $\frac{1}{12}ad^2$, is as the ſtrength of the beam BAC.

To find the ſtrength of the remaining beam BDIC, put $KH=h$, $DI=b$ and $LM=(ah-av+bv)\div h$ $=y$;

$=y$; Then $y\dot{v}^2\dot{v}\div h=(ahv^2\dot{v}-av^3\dot{v}+bv^3\dot{v})\div h^2$; and its fluent (when $v=h$) is $\frac{1}{12}(ah^2+3bh^2)$.

Cor. 1. If $a=d=9$, then $b=1$ and $h=8$; alſo $\frac{1}{12}ad^2=60\cdot75$, and $\frac{1}{12}(ah^2+3bh^2)=64$; that is, the ſtrength of the whole beam is to that of the part (when 1-9th of the depth AK is cut away,) as $60\frac{3}{4}$ to 64.

Cor. 2. If $AK=x$, then $DI=ax\div d$, and the latter expreſſion then becomes $\frac{1}{12}((a+3ax\div d)\times \overline{d-x}|^2)$, the fluxion of which gives $x=\frac{1}{9}$, which ſhews that the ſtrength of the remaining beam is a *maximum*.

Cor. 3. If this laſt expreſſion for the maximum be made equal to $\frac{1}{12}ad^2$, we get $x^2-\frac{5}{3}dx=\frac{1}{3}d^2$; whence $x=2\cdot091673=AK$, when the ſtrength of the remaining beam is equal to that of the whole.

The anſwer by Mr. Lowry *is to the ſame effect.*

The ſame otherwiſe by Mr. John Surtées.

Demit the perpendiculars IS, DQ and let h be $=$ to the height and $2h=$ the baſe, and $x=IS$ any variable height from the baſe: Then (by Emerſon's Fluxions, page 344,) the ſtrength of the parallelogram SIDQ is as $\frac{1}{3}(2hx^2-2x^3)$, and the ſtrength of the triangle CIS ($=BDQ$) is as $\frac{1}{12}x^3$; therefore the ſtrength of the whole trapeziod IDBC is as $\frac{1}{3}(2hx^2-2x^3)+\frac{1}{6}x^3$, which in the preſent caſe ought to be a *maximum*, or $4hx^2-3x^3=$ a *max.* Fluxed and reduced gives $x=\frac{8}{9}h$, which proves the truth of what Emerſon aſſerts.

And thus the anſwer is given by Meſſrs. Bulmer *and* Thornoby.

XX. QUESTION 48, *anſwered by Mr.* G. Sanderſon, London.

If on a as a diameter a circle be deſcribed, and any diſtance (repreſented by x) be taken on the diameter

diameter produced, and a tangent drawn to the circle at that distance be called y; then by the property of the circle $y^2 = (a+x) \cdot x$, or $\frac{y^2}{x} = a + x$.

If x be made the abcissa, and y a perpendicular ordinate, the locus is an *equilateral hyperbola*, whose diameter is a; and $y^2 = (a+x) \cdot x$, or $\frac{y^2}{x} = a + x$.

When $x = 0$, $y = 0$, and $\frac{y^2}{x} = \frac{0}{0} = a + 0 = a$.

As Mathematicians are well acquainted with the usefulness of the expression $\frac{0}{0}$, in determining the limits of curves, I shall give myself no further trouble at present to convince the dabblers in science, such as *Search*, *No Conjurer*, &c. &c. but defer the remainder of my observations 'till I send the solution to the 19th question, in Number III.

Ingenious solutions were likewise given by Messrs. Elliott, Lowry, Simpson, Swale *and* Thornoby.

ARTICLE XLIII.

MATHEMATICAL QUESTIONS,

(To be answered in Number VI.*)*

I. QUESTION 69, *by Mr.* O. G. Gregory.

AN engineer intending to construct a detached bastion, having the saliant angle and the angle of the shoulders each 110°, also the length of each face 60 yards; (measuring along the interior edge of the superior talus of the parapet:) wishes to know what must be the length of each flank, that the gorge may measure no more than 80 yards?

II. QUESTION 70, *by Mr.* Lightfoot, Pupil to Mr. Lowry, at Birmingham.

Standing at the diſtance of 50 feet from the ſide of a river, on the oppoſite bank of which ſtands a ſpire, whoſe top, when my eye was on a level with its bottom, ſubtended an angle of 40° 30′, but my eye being elevated five feet and a half above that level, I found the angle ſubtended by the top of the ſpire, and a mark on the neareſt bank of the river (in a direct line with me and the ſpire) to be 50°40′. From hence I demand the breadth of the river and the height of the ſpire?

III. QUESTION 71, *by Mr.* William Peacock, Land Surveyor, at Birmingham.

A poor, but induſtrious cottager has an acre of land, on an adjoining common, given him by the gentlemen of the pariſh where he reſides, which by agreement is to be in the form of a right parabola; now the poor man not being able to fence it without the aſſiſtance of his well-diſpoſed neighbours, wiſhes ſome ingenious gentleman would tell him what it will coſt at 4d. per yard, and he alſo requeſts a plan of his little field, ſo that the perimeter may be the leaſt poſſible?

IV. QUESTION 72, *by Mr.* Johnſton, Birmingham.

To find three numbers in harmonical proportion, ſuch, that the difference of every two of them may be a ſquare number?

V. QUESTION 73, *by Mr.* Ralph Simpſon.

Having a ſemi-ſpherical veſſel, whoſe diameter is 48 inches, filled with water, I immerſed therein a leaden cylinder whoſe length was 32 inches, and found that the quantity of water overflowing was a *maximum*. What was the diameter of the cylinder's baſe?

VI.

VI. QUESTION 74, *by Mr.* J. H. Swale.

In the periphery of a given circle ABD, are three given points A, B, D: and without the circle a given point P. From A, B two bodies a, b move (in the circumference of the circle,) towards D, but in contrary directions, with celerities which are as m to n; at the same time another body p sets off from P, in a direct line to D, with a given celerity r: it is required to determine the position of the bodies a, b, p, with the distances gone over, when the angle under which a and b are seen from p is a *maximum*.

VII. QUESTION 75, *by Mr.* J. H. Swale.

A ball let fall upon a perfectly inflexible and even plane, from a given height a, at the end of the third reflection ascended to two-thirds of the height it first fell from; required from hence the ratio of its elasticity to perfect elasticity?

VIII. QUESTION 76, *by Mr.* M. A. Harrison.

A piece of dry oak in the form of a conic frustrum whose diameters are 4 and 2 inches and perpendicular height 8 inches, is suspended by a chain from the middle of its side; required the specific gravity of a similar solid, which being cemented to it, the whole may remain in equilibrium, that is, with its axis parallel to the horizon?

IX. QUESTION 77, *by the Rev. Mr.* L. Evans.

In a certain north latitude in the spring of the year 1792, when the sun's declination was double his altitude at six; the difference of the sines of his meridian altitude and mid-night depression was equal to the sine of half the latitude.—*Query*, the time and place?

X. QUESTION 78, *by Mr.* W. Lover, Hampton.

Near the venerable city of Oxford once stood an ancient monument, (in form of a cylinder,) which in one of the stormy blasts of November was blown from its perpendicular direction, so as to make an angle of 70° with the horizon; in this position it remained for several years, 'till the ruinous hand of time had mouldered its foundation, when in five seconds it fell to the ground. Tell me ingenious philomaths how high it was?

XI. QUESTION 79, *by Mr.* Samuel Thornoby.

If from two given points A and B, equi-distant from, but on contrary sides of a right line DE given in position, two lines be drawn to cut it in K and L, and intersect each other in some point F of another right line FE given in position, and if from L a line be drawn to a third given point C, CF be joined, and a line drawn from K parallel to AC and cutting CF in S; the locus of the point S is required?

XII. QUESTION 80, *by Mr.* Thomas Keith.

Given the sun's declination, two altitudes of the sun, taken on the same day, and the time elapsed between the observations, to determine the latitude of the place, by an orthographical construction of the problem, without drawing ellipses?

XIII. QUESTION 81, *by Mr.* A. Buchanan.

The ends of a string 5 feet long are fastened to two immoveable tacks 3 feet asunder, placed in a right line, making with the horizon an angle of 45°. It is required to determine the position of the string when a ring of heavy metal sliding freely thereon rests in equilibrio?

XIV. QUESTION 82, *by Mr.* Richard Elliott.

ABC is half a given segment of a circle, and AED is a quadrant of another given circle; it is required to

draw a radius AL, so that the part ML, intercept-
. between the peripheries of the segment and quad-
nt may be equal to the perpendicular MN let fall
ι AC from the point M in the segment?

XV. QUESTION 83, *by Mr.* John Surtees.

Given the two legs of a right angled spherical
angle, to construct it so as to have the hypo-
enuse on the primitive?

XVI. QUESTION 84, *by Mr.* Swale.

Given the line bisecting the vertical angle and
rminating in the base, the sum or difference of the
eater side and the adjacent segment of the base
ade by that line, and the difference of the angles
the base, to construct the plane triangle?

XVII. QUESTION 86, *by Mr.* John Lowry.

Given the sum of the sides, the vertical angle,
ıd the difference of the angles at the base of a
herical triangle, to project it?

XVIII. QUESTION 86, *by Mr.* Lowry.

Given the vertical angle and the line bisecting it
hen produced to meet the circumscribing circle,
construct the plane triangle, when the sum of
ıe cubes on the two sides is equal to a given cube?

XIX. QUESTION 87, *by Mr.* Lowry.

To determine geometrically, in the diagonal of a
ven square, a point, such, that the sum of the cubes
pon the perpendiculars demitted from that point to
ıe sides of the square may be equal to a given cube?

X. PRIZE QUESTION 88, *by Mr.* William Wallace, *Assistant Teacher of the Mathematics, in the Academy, at Perth.*

If three straight lines touch a parabola, a circle
scribed through their intersections shall pass
rough the focus of the parabola. Required the
emonstration?

ARTICLE XLIV.

Landen's Theorems for the Calculation of Fluents.

(Continued from page 178.)

TABLE IV.

Note. The neceffary explanation refpecting the values of the quantities concerned in the following Theorems is given at the end.

THEOREM I.

$$\dot{F} = x^{-\frac{2}{3}}\dot{x} \div (a^2 - x^2)^{\frac{1}{2}} = \tfrac{3}{2} y^{-\frac{1}{2}}\dot{y} \div (b^3 - y^3)^{\frac{1}{2}}.$$

$$F = K - \tfrac{3}{2} a^{-\frac{2}{3}} B.$$

$$\text{Here } z = \left(a^{\frac{2}{3}} - x^{\frac{2}{3}}\right) \div x^{\frac{2}{3}} = (b - y) \div y.$$

THEOREM II.

The *wh.* flu. $x^{-\frac{2}{3}}\dot{x} \div (a^2 - x^2)^{\frac{1}{2}}$ is $= 3^{\frac{3}{4}} a^{-\frac{2}{3}} P \div (3^{\frac{1}{2}} + 1)$.

THEOREM III.

$$\dot{F} = x^{-\frac{1}{3}}\dot{x} \div (a^2 - x^2)^{\frac{1}{2}} = \tfrac{3}{2}\dot{y} \div (b^3 - y^3)^{\frac{1}{2}}.$$

$$= K - \tfrac{3}{2} a^{-\frac{1}{2}} D.$$

Here $z = (a^{\frac{2}{3}} - x^{\frac{2}{3}}) \div a^{\frac{2}{3}} = (b - y) \div b.$

THEOREM IV.

The *wh.* flu. $x^{-\frac{1}{3}}\dot{x} \div (a^2 - x^2)^{\frac{1}{2}}$ is $= 3^{\frac{1}{4}} a^{-\frac{1}{2}} \text{P} \div (3^{\frac{1}{2}} + 1).$

THEOREM V.

$\dot{\text{F}} = x^{\frac{1}{3}}\dot{x} \div (a^2 - x^2)^{\frac{1}{2}} = \frac{3}{2} y\dot{y} \div (b^3 - y^3)^{\frac{1}{2}}.$

$\text{F} = \text{K} + \frac{3}{2} a^{\frac{1}{3}} \times (\text{C} - \text{D}).$

Here $z = (a^{\frac{2}{3}} - x^{\frac{2}{3}}) \div a^{\frac{2}{3}} = (b - y) \div b.$

THEOREM VI.

The *whole* fluent of $x^{\frac{1}{3}}\dot{x} \div (a^2 - x^2)^{\frac{1}{2}}$ is $= 3^{\frac{1}{4}} a^{\frac{1}{3}} \text{Q}.$

THEOREM VII.

$\dot{\text{F}} = x^{\frac{2}{3}}\dot{x} \div (a^2 - x^2)^{\frac{1}{2}} = \frac{3}{2} y^{\frac{3}{2}} \dot{y} \div (b^3 - y^3)^{\frac{1}{2}}.$

$\text{F} = \text{K} + \frac{1}{4} a^{\frac{2}{3}} \times (\text{A} + \text{B}) - \frac{3}{2} x^{-\frac{1}{3}} \sqrt{(a^2 - x^2)}.$

Here $z = \left(a^{\frac{2}{3}} - x^{\frac{2}{3}}\right) \div x^{\frac{2}{3}} = (b - y) \div y.$

THEOREM

THEOREM VIII.

The *whole* fluent of $x^{\frac{2}{3}}\dot{x}\div(a^2-x^2)^{\frac{1}{2}}$ is $=3^{\frac{1}{4}}a^{\frac{2}{3}}Q\div 2$.

THEOREM IX.

$\dot{F}=x^{-\frac{2}{3}}\dot{x}\div(x^2-a^2)^{\frac{1}{2}}=\frac{3}{2}y^{-\frac{1}{2}}\dot{y}\div(y^3-b^3)^{\frac{1}{2}}$.

$F=K+\frac{3}{2}a^{-\frac{2}{3}}D$.

Here $z=\left(x^{\frac{2}{3}}-a^{\frac{2}{3}}\right)\div x^{\frac{2}{3}}=(y-b)\div y$.

THEOREM X.

The *wh.* flu. $x^{-\frac{2}{3}}\dot{x}\div(x^2-a^2)^{\frac{1}{2}}$ is $=3^{\frac{1}{4}}a^{-\frac{2}{3}}P\div(3^{\frac{1}{2}}+1)$.

THEOREM XI.

$\dot{F}=x^{-\frac{1}{3}}\dot{x}\div(x^2-a^2)^{\frac{1}{2}}=\frac{3}{2}\dot{y}\div(y^3-b^3)^{\frac{1}{2}}$.

$F=K+\frac{3}{2}a^{-\frac{1}{3}}B$.

Here $z=\left(x^{\frac{2}{3}}-a^{\frac{2}{3}}\right)\div a^{\frac{2}{3}}=(y-b)\div b$.

THEO-

THEOREM XII.

The *wh.* flu. $x^{-\frac{1}{3}}\dot{x}\div(x^2-a^2)^{\frac{1}{2}}$ is $=3^{\frac{3}{4}}a^{-\frac{1}{3}}P\div(3^{-\frac{1}{2}}+1)$.

THEOREM XIII.

$\dot{F}=x^{\frac{1}{3}}\dot{x}\div(x^2-a^2)^{\frac{1}{2}}=\frac{3}{2}y\dot{y}\div(y^3-b^3)^{\frac{1}{2}}$.

$F=K+\frac{3}{2}a^{\frac{1}{3}}\times(A+B)$.

Here $z=\left(x^{\frac{2}{3}}-a^{\frac{2}{3}}\right)\div a^{\frac{2}{3}}=(y-b)\div b$.

Note. The *whole* fluent is infinite.

THEOREM XIV.

$\dot{F}=x^{\frac{2}{3}}\dot{x}\div(x^2-a^2)^{\frac{1}{2}}=\frac{3}{2}y^{\frac{3}{2}}\dot{y}\div(y^3-b^3)^{\frac{1}{2}}$.

$F=K+\frac{3}{4}a^{\frac{2}{3}}\times(C-D)+\frac{3}{2}x^{-\frac{1}{3}}\sqrt{(x^2-a^2)}$.

Here $z=\left(x^{\frac{2}{3}}-a^{\frac{2}{3}}\right)\div x^{\frac{2}{3}}=(y-b)\div y$.

Note. The *whole* fluent is infinite:

ARTICLE XLV.

Atwood's *Investigations on Watch Balances.*

(Continued from page 164.)

SINCE watches and time-keepers are uſually adjuſted to mean time when the balance makes 5 vibrations in a ſecond, the time of a ſemivibration will in this caſe $= \frac{1}{10}$ part of a ſecond: the ſubſtitution of $\frac{1}{10}$ for t being made in the preceding equation, the force which accelerates the circumference of the balance, when at any given angular diſtance c° from the quieſcent poſition, will be determined for all time-keepers adjuſted to mean time, in which the balances make 5 vibrations in a ſecond. Suppoſe the given angle $c^{\circ} = 90^{\circ}$; then making $c^{\circ} = 90^{\circ}$, $p = 3.14159$, &c. $l =$ 193, $t = \frac{1}{10}$, the accelerative force at the angular diſtance from quieſcence 90° or $F = p^{2} r\, 90^{\circ} \div (8l/t^{2} \times 180^{\circ}) = r \times 1.00408926$. We have therefore arrived at the following concluſion: if the radius of the balance is equal to one inch, and the time-keeper is adjuſted to mean time when the balance makes 5 vibrations in a ſecond, the force which accelerates the circumference of the balance at the diſtance of 90° from the quieſcent poſition, is $= 1.00408926$, the accelerative force of gravity being $= 1$. And if the radius of the balance is greater or leſs than 1 inch, the force by which the circumference is accelerated at the diſtance of 90° from quieſcence, will be greater or leſs than 1·00408926 in proportion to the radii.

According to the principles aſſumed in the preceding ſolution, the ſpring's elaſtic force is ſuppoſed to vary in the proportion of the angular diſtances from the quieſcent poſition, and on this condition, the vibrations are ſhewn to be iſochronous

ronous, whether they are performed in longer or ſhorter arcs; but if the ſpring's elaſtic force at different diſtances from quieſcence ſhould not be preciſely in the ratio here aſſumed, the longer and ſhorter arcs may be deſcribed in times differing in any proportions of inequality. If, for inſtance, the ſpring's force, inſtead of varying in the ratio of the aforeſaid diſtances, ſhould vary in the $\frac{999}{1000}$ power, $\frac{1001}{1000}$ power of the diſtances, it does not appear from the preceding ſolution what alteration in the daily rate would be cauſed by this change in the law of the force's variation, when the ſemiarc of vibration is increaſed or diminiſhed by a given arc. To aſcertain this point fully, other reſearches will be neceſſary, by which it may be known, what alteration in the daily rate of a time-keeper is occaſioned by a given increaſe or diminution of the arc of vibration, when the ſpring's elaſtic force varies in a ratio of the diſtances from the quieſcent poſition, the general index or exponent of which is any number or fraction n.

The force which accelerates the balance being aſſumed in that power of the diſtances the exponent of which is n, let $BO = b$ (fig. 81, pl. 5.) be the ſemiarc of vibration when the time-keeper is adjuſted to mean time; let $DO = a$; the accelerating force on the circumference at the diſtance from quieſcence $OD = F$; ſuppoſe the circumference to have deſcribed the arc BH from the extremity of the arc B; and let $HO = x$: then the force by which the circumference is accelerated when at the angular diſtance from the quieſcent poſition $OH = Fx^n \div a^n$; let u be the ſpace through which a body falls freely from reſt by the acceleration of gravity, to acquire the velocity of the circumference when it has deſcribed the arc BH; the principles of acceleration give this equa-

 tion

tion: $\dot{u} = -Fx^n\dot{x} \div a^n$: taking the fluents while x decreases from b to n, $u = (Fb^{n+1} - Fx^{n+1}) \div (n+1)\,a^n$, and l being 193 inches, the velocity acquired by the circumference after describing BH, will be $= \sqrt{4lF \div (n+1)a^n} \times \sqrt{b^{n+1} - x^{n+1}}$; let T be the time of describing the arc BH; wherefore $\dot{T} = \sqrt{(n+1)a^n \div 4lF} \times -\dot{x} \div \sqrt{b^{n+1} - x^{n+1}}$. The time of describing the arc BH will be the fluent of this fluxion, while x decreases from b to x, and the time of describing the semiarc BO will be the entire fluent of $\sqrt{(n+1)a^n \div 4lF} \times -\dot{x} \div \sqrt{b^{n+1} - x^{n+1}}$, while x decreases from b to o.

Now let the balance commence its vibration from any other point I, (fig. 81, pl. 5.) and let IO $= c$; suppose the circumference to have described the arc IK, and make OK $= y$; let t be the time of describing the arc IK; then by proceeding in the same manner as in the former case, it is found that $\dot{t} = \sqrt{(n+1)a^n \div 4lF} \times -\dot{y} \div \sqrt{c^{n+1} - y^{n+1}}$; and the time of describing the semiarc IO, will be the entire fluent of this fluxion, while y decreases from c to o. Although the fluents of the fluxions $-\dot{x} \div \sqrt{b^{n+1} - x^{n+1}}$ and $-\dot{y} \div \sqrt{c^{n+1} - y^{n+1}}$ cannot be expressed in general terms, yet the exact proportion of the said fluents may be assigned, which will be the proportion of the times in which the balance vibrates in the two semiarcs BO, IO; the multiplying quantity $\sqrt{(n+1)a^n \div 4lF}$ being common

to

to both fluxions; and ſince the entire fluent * of $-\dot{x} \div \sqrt{b^{n+1} - x^{n+1}}$ is to the entire fluent of $-\dot{y} \div \sqrt{c^{n+1} - y^{n+1}}$ as $b^{\frac{1-n}{2}}$ is to $c^{\frac{1-n}{2}}$, it follows, that the time of a ſemivibration in the arc BO is to the time of a ſemivibration in the arc IO, as $b^{\frac{1-n}{2}}$ to $c^{\frac{1-n}{2}}$ or as 1 to $(IO \div BO)^{\frac{1-n}{2}}$.

ART-

* $\dot{x} \div \sqrt{b^{n+1} - x^{n+1}} = (1 \div b^{\frac{n+1}{2}}) \times \dot{x} \div \sqrt{1 - \overline{x \div b}|^{n+1}}$

and $\dot{y} \div \sqrt{c^{n+1} - y^{n+1}} = (1 \div c^{\frac{n+1}{2}}) \times \dot{y} \div \sqrt{1 - \overline{y \div c}|^{n+1}}$.

To find the proportion of the entire fluent of $\dot{x} \div \sqrt{1 - \overline{x \div b}|^{n+1}}$ to the entire fluent of $\dot{y} \div \sqrt{1 - \overline{y \div c}|^{n+1}}$ make $x = by \div c$, ſo that when $x = 0$, $y = 0$, and when $x = b$, $y = c$; then $\dot{x} \div \sqrt{1 - \overline{x \div b}|^{n \times 1}} = (b \div c) \times \dot{y} \div \sqrt{1 - \overline{y \div c}|^{n+1}}$, and the proportion of $\dot{x} \div \sqrt{1 - \overline{x \div b}|^{n+1}}$ to $\dot{y} \div \sqrt{1 - \overline{y \div c}|^{n+1}}$ will be equal to that of $b \div c$ to 1, or of b to c; this being the conſtant proportion of the fluxions when $x = by \div c$, the fluents will be in the ſame proportion, provided $x = by \div c$; wherefore the entire fluent of $\dot{x} \div \sqrt{b^{n+1} - x^{n+1}}$ will be to the entire fluent of $\dot{y} \div \sqrt{c^{n+1} - x^{n+1}}$ as $b \div b^{\frac{n+1}{2}}$ to $c \div c^{\frac{n+1}{2}}$ or as $b^{\frac{1-n}{2}}$ to $c^{\frac{1-n}{2}}$. It is not neceſſary to add conſtant quantities to the fluents of the fluxions $-\dot{x} \div \sqrt{b^{n+1} - x^{n+1}}$, $-\dot{y} \div \sqrt{c^{n+1} - y^{n+1}}$; becauſe when the entire fluents are taken, they are preciſely in the ſame proportion, whether the conſtant quantities (or corrections as they are ſome times termed) are added or omitted.

ARTICLE XLVI.

On the Resolution of Indeterminate Problems;

By John Leslie, A. M.

(Continued from page 159.)

PROBLEM V.

TO find two squares which, diminished by unit, shall be in a given ratio.

By hypothesis, $a : b :: x^2 - 1 : y^2 - 1$; whence the equation $ay^2 - a = bx^2 - b$, and by resolution, $(ay + a)\ (y - 1) = (b$[illegible]$+ b)$ [illegible] $- 1)$: wherefore by assumption, $ay + a =$ [illegible] and $my - m = bx + b$. Transposing the first, [illegible] $- m - a$, and dividing $y = (mx - m - a) \div$ [illegible] ansposing the second, $my = bx + b + m$, and di[illegible] $y = (bx + b + m) \div m$, wherefore, $(mx - m - a)$[illegible]$(bx + b + m) \div m$, and reducing $m^2x - m^2 - ma$ [illegible]$x + ab + ma$, that is, $m^2x - abx = m^2 + ab + 2ma$, and therefore, $x = (m^2 + ab + 2ma) \div (m^2 - ab)$; but $y = (bx + b + m) \div m$, consequently $y = (m^2 + ab + 2mb) \div (m^2 - ab)$.

Suppose $a = 2$, $b = 3$, and $m = 3$; then $x = (9 + 6 + 12) \div (9 - 6) = 9$, and $y = (9 + 6 + 18) \div (9 - 6) = 11$; but $2 : 3 :: 80 : 120$.

Cor. 1. When the numbers x and y are very great, it is obvious that the ratio of $x^2 - 1$ to $y^2 - 1$, will be nearly equal to that of x^2 to y^2; and consequently the ratio of $\sqrt{a}$ to $\sqrt{b}$ will be still more nearly equal to that of x to y. If a and b, besides be nearly equal, the approximation will be more accurate. Let $m = a$; then the denominator $m^2 - ab$ will be small, and therefore the fractions large; whence, by substitution $\sqrt{a} : \sqrt{b} :: (a^2 + ab + 2a^2) \div (a^2 - ab) : (a^2 + ab + 2ab) \div (a^2 - ab) = 3a^2 + ab : 3ab + a^2 = 3a + b : 3b + a$, nearly.

Thus $\sqrt{49} : \sqrt{50} :: 197 : 199 :: 7 : 7 + \frac{14}{197}$, hence $\sqrt{50} = 7{\cdot}07107$, true to the last place.

Cor. 2. Let $m = \frac{1}{2}(a + b)$; then $m^2 - ab = \overline{\tfrac{1}{2}(a + b)}^2 - ab = \overline{\tfrac{1}{2}(a - b)}^2$, which when a and b are nearly equal,

equal, will be small, and by substitution, $\sqrt{a}$: $\sqrt{b}$ :: $(\frac{1}{2}(a+b))^2+ab+a(a+b)) \div ((\frac{1}{2}(a+b))^2-ab)$: $((\frac{1}{2}(a+b))^2+ab+b(a+b)) \div ((\frac{1}{2}(a+b))^2-ab)$, nearly; hence, by proper reductions $\sqrt{a} : \sqrt{b} :: (5a^2+10ab+b^2) : 5b^2+10ab+a^2$. This formula is more intricate than the former, but still more accurate. Thus, $\sqrt{6} : \sqrt{10} :: 405+900+100 : 500+900+81 = 1405 : 1481$, and $\sqrt{10}=3\cdot16209$, true to the last place.

PROBLEM VI.

Let it be required to find a number, such that, if given multiples of it be increased by given numbers, the product of the sums shall be a square

Let $(ex+f)(gx+h)=y^2$; by assumption $ex+f=my$ and $gx+h=y\div m$. Transposing the first equation, and dividing, $x=(my-f)\div e$. Reducing the second, $mgx+mh=y$, and transposing and dividing, $x=(y-mh)\div mg$; whence, $(my-f)\div e=(y-mh)\div mg$, and reducing, $m^2gy-mfg=ey-meh$, and transposing, $m^2gy-ey=mfg-meh$, and consequently $y=(mfg-meh)\div(m^2g-e)$. Also $x=(y-mh)\div mg=(f-m^2h)\div(m^2g-e)$.

Suppose $(7x+6)(2x+1)=y^2$. If $m=2$, then $x=(6-4)\div(8-7)=2$, and $y=(24-14)\div(8-7)=10$; but $20\times5=100=(10)^2$.

Cor. Let $e=1$ and $g=1$; the hypothesis will become $(x+f)(x+h)=y^2$. In this case, we obtain $x=(f-m^2h)\div m^2-1$, and $y=(mf-mh)\div(m^2-1)$. Thus, if $(x+12)(x+2)$, where $f=12$, and $h=2$, and $m=\frac{3}{2}$; then $x=(12-18\div4)\div(9\div4-1)=6$, and $y=(\frac{3}{2}\times10)\div(9\div4-1)=12$; but $18\times8=144=(12)^2$.

PROBLEM VII.

Let it be required to find rational values of x and y, in the general quadratic $Ax^2+Bx+C=y^2$.

Case

Case 1. When the first term is a square.

Suppose $A=a^2$, when the expression becomes $a^2x^2+bx+c=y^2$; by transposition, $bx+c=y^2-a^2x^2$, and resolving into factors, $b(x+c\div b)=(y+ax)(y-ax)$; whence by assumption $x+c\div b=my-max$, and $b=(y+ax)\div m$. Reducing the first equation, $bx+c=mby-mabx$, and $y=(mabx+bx+c)\div mb$. Again reducing the second, $mb=y+ax$, and $y=mb-ax$; consequently $(mabx+bx+c)\div mb=mb-ax$, or $mabx+bx+c=m^2b^2-mabx$, and therefore, $x=(m^2b^2-c)\div(2mab+b)$. But $y=mb-ax$; therefore, $y=(m^2ab^2+mb^2+ac)\div(2mab+b)$.

Suppose $9x^2+7x+14=y^2$, and $m=2$; then $x=(4\times49-14)\div(4\times21+7)=2$, and $y=(4\times147+2\times49+42)\div(4\times21+7)=8$; but $9\times4+7\times2+14=64=(8)^2$.

Cor. 1. Let $a=1$, the expression becomes $x^2+bx+c=y^2$; and $x=(m^2b^2-c)\div(2mb+b)$, and $y=(m^2b^2+mb^2+c)\div(2mb+b)$. Thus, if $x^2+4x+4=y^2$, and $m=2$; then $x=(64-4)\div(16+4)=3$, and $y=(64+32+4)\div(16+4)=5$; but $9+4\times3+4=25=(5)^2$.

Cor. 2. When the third term is wanting, the expression becomes $a^2x^2+bx=y^2$; and in this case, the formulæ will become by reduction, $x=m^2b\div(2ma+1)$, and $y=(m^2ab+mb)\div(2ma+1)$. Thus, if $9x^2+13x=y^2$, and $m=2$; then $x=52\div(4\times3+1)=4$, and $y=(4\times39+2\times13)\div(4\times3+1)=14$; but $9\times16+4\times13=196=(14)^2$.

ART-

ARTICLE XLVII.

Demonstrations to Dr. Stewart's *Propositions proposed in* ARTICLE XXI.

PROP. XV. THEO. XII. *Fig.* 159, *Plate.* 11.

Demonstrated by Mr. John Lowry.

LET there be any number of right lines AB, AC, AD, AE, &c. given by position intersecting each other in the point A; two right lines AY, AZ may be found which will be given by position, such, that if from any point X there be drawn the perpendiculars XB, XC, XD, XE, &c. to the right lines AB, AC, AD, AE, &c. given by position, and likewise XY, XZ perpendiculars to AY, AZ the two right lines found, twice the sum of the squares of the perpendiculars XB, XC, XD, XE, &c. drawn to the right lines AB, AC, AD, AE, &c. given by position, will be equal to the multiple of the sum of the squares of the perpendiculars XY, XZ drawn to AY, AZ the two right lines found by the number of the lines given by position.

Join AX and bisect it in Q; with the centre Q and distance QX or QA describe a circle intersecting the given lines in the points B, C, D, E, &c. Find the point V as in Prop. IX. for the points B, C, D, E, &c. join VQ and at right angles thereto draw YVZ meeting the circle in Y, Z; join AY, AZ and they will be the two right-lines that was to be found.

Join QB, QC, QD, QE, &c. VB, VC, VD, VE, &c. and VX, QY, QZ; XB, XC, XD, XE, &c. and XY, XZ being joined, will be perpendicular to AB, AC, AD, AE, &c. and AY, AZ respectively

By Prop. IX. the sum of the squares of QB, QC, QD, QE, &c. that is, the multiple of the square

square of the semidiameter QY of the circle by the number of the lines given by position, is equal to the sum of the squares of VB, VC, VD, VE, &c. together with the multiple of the square of VQ by the number of the lines given by position. But the square of the semidiameter QY is equal to the sum of the squares VY, VQ; therefore the multiple of the sum of the squares of VY, VQ by the number of the lines given by position is equal to the sum of the squares of VB, VC, VD, VE, &c. together with the multiple of the square of VQ by the number of the lines given by position, that is, the multiple of the square of VY by the number of the lines given by position, is equal to the sum of the squares of VB, VC, VD, VE, &c.

Again, by Prop. IX. the sum of the squares of XB, XC, XD, XE, &c. is equal to the sum of the squares of VB, VC, VD, VE, &c. together with the multiple of the square of VX by the number of the lines given by position; therefore the sum of the squares of XB, XC, XD, XE, &c. is equal to the multiple of the sum of the squares of VX, VY by the number of the lines given by position. But YZ is bisected in V; therefore (by Prop. II. *Cor.*) the sum of the squares of XY, XZ is equal to twice the sum of the squares of VX, VY; and therefore twice the sum of the squares of XB, XC, XD, XE, &c. is equal to the multiple of the sum of the squares of XY, XZ by the number of the lines given by position.

Note. I presume it is unnecessary to prove that the right lines AY, AZ are given by position, since their position has really been determined in the construction.

The same demonstrated by Mr. J. H. Swale.

Let there be any number of right lines IA, IB, IC, ID, IE, &c. (fig. 160, pl. 11.) given by position intersecting each other in the point I; two right

ght lines QS, RT may be found that will be given y position, such, that if from any point P there e drawn the perpendiculars PA, PB, PC, PD, E, &c. to the right lines IA, IB, IC, ID, IE, &c. iven by position, and likewise PS, PT perpendiculars to QS, RT the two right lines found, twice he sum of the squares of the perpendiculars PA, PB, PC, PD, PE, &c. drawn to the right lines IA, B, IC, ID, IE, &c. given by position will be equal o the multiple of the sum of the squares of the perpendiculars PS, PT drawn to QS, RT the two lines ound by the number of the lines IA, IB, IC, ID. E, &c. given by position.

Let there be five right lines IA, IB, IC, ID, IE, given by position intersecting each other in the point ; from any point P demit the perpendiculars PA, PB, PC, PD, PE, and join AB, CB, DB, EB; ake AF = one-fifth of AB, CG = one-fifth of BC, DH = one-fifth of DB, EK = one-fifth of EB and join PF, PG, PH, PK, FK and HG; bisect KF, BF, GH, BH in L, N, M, V, and join PL, BL, NL, PM, BM, VM; also bisect LF, MH in O, W and join NO, VW: make OU, WY perpendicular to ON, VW, and equal to OF, WH respectively; join NU, VY and perpendicular hereto draw NX, VZ equal to LF, MH respectively, and join XU, YZ; again, take LS, MT perpendicular to PL, PM and equal to XU, YZ and join PS, PT; perpendicular to the lines PS, PT given n position, draw QS, RT given also in position and they will be the lines that was to be found.

For, four times the square of AP together with he square of BP is equal to five times the square of FP together with the rectangle ABF; four times the square of CP together with the square BP is equal to five times the square of GP together with the rectangle CBG, four times the square of DP together with the square of BP is equal to five times

the

the ſquare of HP together with the rectangle DBH, and four times the ſquare of EP together with the ſquare of BP is equal to five times the ſquare of KP together with the rectangle EBK; and therefore, four times the ſum of the ſquares of AP, BP, CP, DP, EP, is equal to five times the ſum of the ſquares of FP, GP, HP, KP, together with the ſum of the rectangles ABF, CBG, DBH, EBK, that is, equal to five times the ſum of the ſquares of FP, GP, HP, KP, together with four times the ſum of the ſquares of AF, CG, DH EK, together with the ſum of the ſquares of FB, GB, HB, KB, that is, equal to five times the ſum of the ſquares of FP, GP, HP, KP, LN, NF, MV, VH. But, five times the ſum of the ſquares of FP, KP, is equal to ten times the ſum of the ſquares of PL, LF; five times the ſum of the ſquares of GP, HP, is equal to ten times the ſum of the ſquares of PM, GM; five times the ſum of the ſum of the ſquares of LN, NF is equal to ten times the ſum of the ſquares of NO, NF, and five times the ſum of the ſquares of MV, VH, is equal to ten times the ſum of the ſquares of VW, WH; therefore twice the ſum of the ſquares of AP, BP, CP, DP, EP, is equal to five times the ſum of the ſquares PL, LF, NO, OF, PM, GM, VW, WH. But, five times the ſum of the ſquares of NO, OF (UO), LF (NX), is equal to five times the ſquare of LS; five times the ſum of the ſquares VW, WH (WY), MG (VZ), is equal to five times the ſquare of MT. Wherefore twice the ſum of the ſquares of AP, BP, CP, DP, EP, is equal to five times the ſum of the ſquares of PL, LS, PM, MT, that is, equal to five times the ſum of the ſquares of PS, PT.

The

The same demonstrated by Dr. Small.

LEMMA III. Added by Dr. Small.

Let there be a figure ABCD (fig. 132, pl. 9.) given in species inscribed in a circle, the straight line EH drawn from E, the centre of the circle, to H, the centre of gravity of the figure, will have a given ratio to the semidiameter, and will make given angles with the semidiameters, drawn to the angular points of the figure.

The centre of gravity of the figure ABCD is found by bisecting AB in F, by joining FC and dividing it in G, so that CG=2GF, and by joining GD and dividing it in H, so that DH=3HG. Hence, and by joining BD and CA, the lemma will be manifest.

For the triangle BFE is right-angled in F, and the angle BEF=ADB, is given. Therefore the ratio of BE, or CE, to EF is given.

Again, in the triangle CEF, the angle CEF =BEC+BEF=2BDC+ADB= a given angle; and since the ratio of CE to EF, and of CG to GF are given, the line EG will divide the triangle CFE into two triangles given in species. Therefore the angle CEG, and the ratio of CE or DE, to EG, are given.

Lastly, in the triangle DEG; the angle DEG =2DAC+CEG, is given; and since the ratio of DE to EG, and of DH to HG, are given, the line EH will divide the triangle DEG into two triangles given in species. Therefore the angle DEH, and the ratio of DE to EH will be also given.

Let there be any number, m, of straight lines AB, AC, AD, AE, &c. (fig. 133, pl. 9.) given by position, intersecting one another in the point A, two straight lines AX, AY, may be found, which will be given by position, such, that if

from any point F there be drawn the perpendiculars FB, FC, FD, FE, &c. to AB, AC, AD, AE, &c. and FX, FY, perpendicular to AX, AY, $2(FB^2+FC^2+FD^2+FE^2+\&c.)=m(FX^2+FY^2)$.

Let m be $=4$. Let G be the centre of the circle which passes through A, B, C, D, E, F, and H the centre of gravity of the figure BCDE. Join GH, and through H draw HXY perpendicular to GH, meeting the circumference in X, Y, and join GB, GC, GD, GE; HB, HC, HD, HF, AX, AY, FX, FY. Then, by Theorem 6. $GB^2+GC^2+GD^2+GE^2=4GB^2=HB^2+HC^2+HD^2+HE^2+4HG^2$.
But $4GB^2=4GX^2=4(GH^2+HX^2)$. Theref. also $HB^2+HC^2+HD^2+HE^2+4HG^2=4(HG^2+HX^2)$; or, $HB^2+HC^2+HD^2+HE^2=4HX^2$. Again, by Theorem 6. $FB^2+FC^2+FD^2+FE^2=HB^2+HC^2+HD^2+HE^2+4FH^2$, and therefore $FB^2+FC^2+FD^2+FE^2=4(FH^2+HX^2)$. That is, $2(FB^2+FC^2+FD^2+FE^2)=8(FH^2+HX^2)=4(FX^2+FY^2)$, (Prop. 1.).

But because the lines AB, AC, AD, AE, are given by position, the angles BAC, CAD, DAE, BAE, are given; therefore the angles BGC, CGD, DGE, BGE, which are the doubles of them, are also given, and the isosceles triangles BGC, CGD, DGE, BGE are given in species. Consequently, the ratio of the semidiameter GB to each of the lines BC, CD, DE, BE, is given, and therefore the ratios of BC, CD, DE, BE, to one another, are given; and the angles of the figure BCDE are also given, therefore the figure itself is given in species. Therefore (Lemma 3.) the ratio of GX to GH is given; and since the angle GHX is a right angle, the triangle GHX is given in species. therefore XGH, YGH are given. But BGH is given, (Lemma 3.); therefore BGX, BGY, and their halves BAX, BAY, are also given; and since BA is given by position, and the point A, the lines AX, AY, are also given by position.

But

But FX, FY, are perpendicular to AX, AY, and it has been ſhewn that $2(FB^2+FC^2+FD^2+FE^2)=4(FX^2+FY^2)$. Therefore AX, AY are the two lines required to be found.

The conſtruction is obvious, by aſſuming a point F, which, for the greater ſimplicity, may be in one of the given lines, and by deſcribing the figure as above.

Cor. I. Added by Mr. Lowry.

Let there be any number of right lines given by poſition, interſecting each other in a point; any even number of right lines may be found which will be given by poſition, ſuch, that if from any point there be drawn perpendiculars to all the right lines given by poſition, and likewiſe to all the right lines found, the multiple of the ſum of the ſquares of the perpendiculars drawn to the right lines given by poſition, by the number of the right lines found, will be equal to the multiple of the ſum of the ſquares of the perpendiculars drawn to the right lines found by the number of the lines given by poſition.

Cor. II. Added by Mr. Lowry.

Let there be any number of circles given by poſition, having the ſame centre; two circles may be found, which will be given by poſition, ſuch, that if from any point without the circles there be drawn tangents to all the circles given by poſition, and likewiſe to the two circles found, twice the ſum of the ſquares of the tangents drawn to the circles given by poſition will be equal to the multiple of the ſum of the ſquares of the tangents drawn to the two circles found by the number of the circles given by poſition.

Cor. III. Added by Dr. Small.

If from any point parallels be drawn to AB, AC, AD, AE, and to AX, AY, cutting the perpendiculars FB, FC, FD, FE, and FX, FY, in b, c, d, e, and in x, y,
$$2(Fb^2 + Fc^2 + Fd^2 + Fe^2) = 4(Fx^2 + Fy^2).$$

LEMMA IV. Added by Dr. Small.

Let AB, AC, (fig. 134, pl. 9.) be two ſtraight lines given by poſition, interſecting one another in the point A, and from any point D let DB, DC, be drawn perpendicular to AB, AC; let CB be joined and biſected in E, and from E let EF be drawn parallel, and equal to a given ſtraight line; through F let GFH be drawn to meet DB and DC, ſo as to be biſected in F, and through G and H, let GK, HK, be drawn parallel to AB, AC: the lines GH, HK will be given by poſition. Through F draw LM parallel to BC, and through B and C draw BL and CM parallel to EF; join GL, HM; from A draw AN parallel and equal to EF; join LN, MN; through N draw OP parallel to GL, and join AO, AP.

Becauſe GF=FH, and LF=FM, GL will be equal and parallel to HM; and becauſe AN is equal and parallel to BL and to CM, the figures AM and AL are parallelograms. Therefore NL is parallel to GK, and NM to HK. Therefore NG and NH are parallelograms, and OG=NL =AB; hence AO is perpendicular to GK; and in the ſame manner, AP is perpendicular to HK. Therefore NO=LG=HM=NP. But the angle AOP is given, being the ſupplement of OKP, and ſince the point N is given, and NO=NP, the points O and P are given; and therefore AO and AP. Therefore the lines GK, HK, are given by poſition.

PROP.

PROP. XVI. THEO. XIII. Fig. 161, Pl 11.

Demonstrated by Mr. John Lowry.

Let there be any number of right lines AB, BC, CD, DA, &c. given by position, that are neither all parallel, nor intersecting each other in one point; two right lines PY, PZ, may be found, that will be given by position, such, that if from any point X there be drawn perpendiculars XE, XF, XG, XH, &c. to the right lines AB, BC, CD, DA, &c. given by position, and likewise XY, XZ, perpendiculars to PY, PZ, the two right lines found, twice the sum of the squares of the perpendiculars XE, XF, XG, XH, &c. drawn to the right lines AB, BC, CD, DA, &c. given by position, will be equal to the multiple of the sum of the squares of the perpendiculars XY, XZ, drawn to PY, PZ, the two right lines found by the number of the right lines given by position, together with a given space.

Find the point Q as in Prop. IX. for the points E, F, G, H, &c. from the point A, where two of the lines given by position intersect each other, draw AI, AK, &c. parallel to the rest of the lines BC, CD, &c. given by position. Let two right lines AL, AR, be found as in Prop. XV. for the right lines AI, AB, AD, AK, &c. and the point X. Draw XR, XL perpendicular to AR. AL, and through the point Q draw SQT perpendicular to XL; take QT equal to QS, and draw TY parallel to SX, meeting XR, produced, if necessary, in Y. Draw YQZ meeting XL in Z; then if ZP, YP, be drawn parallel to AL, AR, and intersecting each other in P, they will be two such right lines as are required.

From P draw Pa, Pb, Pc, Pd, &c. parallel to the right lines AB, BC, CD, DA, &c. given by position,

ſition, meeting the perpendiculars XE, XF, XG, XH, &c. in a, b, c, d, &c. Join QE, QF, QG, QH, &c. PE, PF, PG, PH, &c. PX, PQ, and QX; and from P let PM, PN, PO, PV, &c. be drawn perpendicular to the right lines AB, BC, CD, DA, &c. given by poſition.

By Prop. IX. the ſum of the ſquares of PE, PF, PG, PH, &c. is equal to the ſum of the ſquares of QE, QF, QG, QH, &c. together with the multiple of the ſquare of PQ by the number of the lines given by poſition. But the ſquare of PE is equal to the ſum of the ſquares of PM, ME, that is, equal to the ſum of the ſquares of PM, Pa, and in the ſame way it will appear that the ſquare of PF is equal to the ſum of the ſquares of PN, Pb; that the ſquare of PG is equal to the ſum of the ſquares of PO, Pc; that the ſquare of PH is equal to the ſum of the ſquares PV, Pd; and ſo on. Therefore the ſum of the ſquares QE, QF, QG, QH, &c. together with the multiple of the ſquare of PQ, by the number of the lines given by poſition, is equal to the ſum of the ſquares of PM, PN, PO, PV, &c. together with the ſum of the ſquares of Pa, Pb, Pc, Pd, &c.

Again (Prop. XV.) twice the ſum of the ſquares of XE, XI, XK, XH, &c. is equal to the multiple of the ſum of the ſquares of XR, XL, by the number of the right lines given by poſition. But becauſe Pa, Pb, Pc, Pd, &c. are drawn from the point P, parallel to AB, AI, AK, AD, &c. and PY, PZ, are likewiſe drawn parallel to AR, AL; therefore it is eaſily ſhewn (Prop. XV.) that twice the ſum of the ſquares of Xa, Xb, Xc, Xd, &c. is equal the multiple of the ſum of the ſquares of XY, XZ, by the number of the right lines given by poſition. But the multiple of the ſquare of PX, by the number of the right lines given by poſition, is equal to the ſum of the ſquares of Xa, Xb, Xc. Xd, &c. together

gether with the ſum of the ſquares of Pa, Pb, Pc, Pd, &c. and twice the multiple of the ſquare of PX, by the number of the right lines given by poſition, is equal to the multiple of the ſum of the ſquares of XY, XZ, by the number of the right lines given by poſition, together with the multiple of the ſum of the ſquares of PY, PZ, by the ſame number; therefore twice the ſum of the ſquares of Pa, Pb, Pc, Pd, &c. is equal to the multiple of the ſum of the ſquares of PY, PZ, by the number of the right lines given by poſition. But it has been ſhewn that the ſum of the ſquares of QE, QF, QG, QH, &c. together with the multiple of the ſquare of PQ, by the number of the right lines given by poſition, is equal to the ſum of the ſquares of PM, PN, PO, PV, &c. together with the ſum of the ſquares of Pa, Pb, Pc, Pd, &c. therefore twice the ſum of the ſquares of QE, QF, QG, QH, &c. together with twice the multiple of the ſquare of PQ, by the number of the right lines given by poſition, is equal to twice the ſum of the ſquares of PM, PN, PO, PV, &c. together with twice the multiple of the ſum of the ſquares of PY, PZ, by the number of the right lines given by poſition.

Again, (Prop. IX.) the ſum of the ſquares of QE, QF, QG, QH, &c. together with the multiple of the ſquare of XQ, by the number of the lines given by poſition, is equal to the ſum of the ſquares of XE, XF, XG, XH, &c. therefore twice the ſum of the ſquares of XE, XF, XG, XH, &c. is equal to twice the ſum of the ſquares of PM, PN, PO, PV, &c. together with the difference between twice the multiple of the ſum of the ſquares of PY, PZ, XQ, by the number of the right lines given by poſition, and twice the multiple of the ſquare of PQ by the ſame number.

But TQ is equal to SQ, and TY is parallel to SX; therefore QY is equal to QZ: therefore (Prop. II.) the ſum of the ſquares of PY, PZ, is equal

equal to twice the ſum of the ſquares of PQ, QY; therefore the difference between the ſum of the ſquares of PY, PZ, XQ, and the ſquare of PQ, is equal to twice the ſum of the ſquares XQ, YQ, that is, (Prop. II.) equal to the ſum of the ſquares of XY, XZ. Becauſe the point P has been determined in the conſtruction, the lines PM, PN, PO, PV, &c. are given in magnitude, therefore the ſum of their ſquares is equal to a given ſpace. Therefore twice the ſum of the ſquares of XE, XF, XG, XH, &c. is equal to the multiple of the ſum of the ſquares of XY, XZ, by the number of the right lines given by poſition, together with a given ſpace.

The ſame demonſtrated by Dr. Small, Fig. 135, Pl. 9.

Let there be any number, m, of ſtraight lines AB, BC, CD, DA, &c. given by poſition, neither all parallel nor interſecting in one point, two ſtraight lines, XY, XZ, may be found, which will be given by poſition, ſuch, that if from any point E there be drawn perpendiculars EF, EG, EH, EK, &c. to AB, BC, CD, DA, &c. and EY, EZ perpendiculars to XY, XZ.
$2(EF^2+EG^2+EH^2+EK^2 \&c.)=m(EY^2+EZ^2)+A^2$, A^2 being a given ſpace.

Let $m=4$, and from C, one of the points of interſection, draw Cf, Ck, parallel to the lines given by poſition that do not interſect in C. Let two ſtraight lines CL, CM be found, ſuch, that $2(Ef^2+EG^2+EH^2+Ek^2)=4(EL^2+EM^2)$, (Theor. 12.). Let N be the centre of gravity of the four points F, G, H, K, (Theor. 6.). Through N draw YNZ, to meet EL, EM in Y, Z, and ſo as to be biſected in N. Through Y and Z draw YX, ZX perpendicular to EL, EM, interſecting each other in X. From X draw XP, XQ, XR, XS perpendicular, and Xa, Xb, Xc, Xd, parallel to AB, BC, CD, DA; let Xa, Xb, Xc, Xd meet EF, EG, EH, EK in a, b, c, d;

c, d; and let O be the centre of gravity of the four points f, G, H, k, where the parallels from C, to the lines given by position, meet the perpendiculars from E.

By Theor. 6. $2(XF^2+XG^2+XH^2+XK^2)=2(NF^2+NG^2+NH^2+NK^2)+8NX^2$. But $2(XF^2+XG^2+XH^2+XK^2)=2(XP^2+XQ^2+XR^2+XS^2)+2(Xa^2+Xb^2+Xc^2+Xd^2)$. Therefore $2(NF^2+NG^2+NH^2+NK^2)+8NX^2=2(XP^2+XQ^2+XR^2+XS^2)+2(Xa^2+Xb^2+Xc^2+Xd^2)$. But since, $2(Ef^2+EG^2+EH^2+Ek^2)=4(EL^2+EM^2)$, and from the point X parallels to Cf, CG, CH, Ck, and to CL, CM, are drawn, cutting the perpendiculars from E to these lines, in a, b, c, d, and in Y, Z, therefore, by Cor. Theor. 12. $2(Ea^2+Eb^2+Ec^2+Ed^2)=4(EY^2+EZ^2)$, and consequently $2(Xa^2+Xb^2+Xc^2+Xd^2)=4(XY^2+XZ^2)=8(NY^2+NX^2)$, (Prop. 1.) Therefore

$2(NF^2+NG^2+NH^2+NK^2)=2(XP^2+XQ^2+XR^2+XS^2)+8NY^2$. But by Theor. 6.

$2(EF^2+EG^2+EH^2+EK^2)=2(NF^2+NG^2+NH^2+NK^2)+8NE^2$. Therefore

$2(EF^2+EG^2+EH^2+EK^2)=2(XP^2+XQ^2+XR^2+XS^2)+8(NY^2+NE^2)$; or,

$2(EF^2+EG^2+EH^2+EK^2)=2(XP^2+XQ^2+XR^2+XS^2)+4(EY^2+EZ^2)$, (Prop. 1.)

It remains to demonstrate that X is a given point, and that XY, XZ, are lines given in position.

The point O may be found, by bisecting (fig. 136, pl. 9.) GH in g, joining gk, and dividing it in m, so that $gm=\frac{1}{3}gk$, and joining fm, and dividing it in O, so that $mO=\frac{1}{4}mf$; and in the same manner the point N may be found by joining gK and making $gn=\frac{1}{3}gK$, and joining nF, and making $nN=\frac{1}{4}nF$; let mn be joined, through O draw Op, and through N draw Nq, both parallel to EF, and meeting mn in p, q; let EF meet mn in r, join ON, and through O draw Os parallel to mn, meeting Nq in s.

Then

Then becauſe $gm\frac{1}{3} = gk$, and $gn = \frac{1}{3}gK$, the line mn is parallel and equal to $\frac{1}{3}Kk$. Becauſe alſo $Nn = \frac{1}{4}Fn$, $Nq = \frac{1}{4}Fr$; and for the ſame reaſon $OP = \frac{1}{4}fr$. Therefore $pq = Os = \frac{3}{4}mn = \frac{1}{4}Kk$. But the angle OsN is given, for it is equal to kEF; and ſince Os is given, and $Ns = Nq - sq$, NO is alſo given. But (fig. 135, pl. 9.) ſince the lines CL, CM, interſecting in the point C, are given by poſition, and from the point E there are drawn to them the perpendiculars EL, EM, and LM is joined, and biſected in O, and from O there is drawn a ſtraight line ON, given both by poſition and magnitude, and YNZ is drawn through N to meet EL, EM in Y, Z, and ſo as to be biſected in N, and from Y and Z, YX, ZX are drawn parallel to CL, CM; therefore, by Lemma 4. YX, ZX are given by poſition; and conſequently the point X of their interſection is given, and therefore alſo XP, XQ, XR, XS. But EY, EZ are perpendicular to XY, ZX; and it has been proved that $2(EF^2+EG^2+EH^2+EK^2) = 4(EY^2+EZ^2)+2(XP^2+XQ^2+XR^2+XS^2)$, and theſe four laſt ſquares are given. Therefore XY, XZ, are the two lines required to be found, and $2(EF^2+EG^2+EH^2+EK^2) = 4(EY^2+EZ^2)+A^2$.

The point X, found in this propoſition, is the centre of gravity of the four points P, Q, R, S, where perpendiculars, drawn from it, meet the four lines given by poſition. It is alſo a point, ſuch, that the ſum of the ſquares of the perpendiculars drawn from it, to the lines given by poſition, is a minimum.

Cor. Demonſtrated by Dr. Small.

If the ſtraight lines (fig. 163, pl. 11.) AB, BC, CA be ſo ſituated as to form an equilater
about a circle, or a ſemicircle; or if the num
lines given by poſition be even, and every
two

two interfect each other at right angles, the two lines XY, XZ, that may be found, will interfect each other at right angles.

Let the lines AB, BC, CA, that are given by pofition, form an equilateral triangle. Let X be the point in that triangle, which is the centre of gravity of the three points K, L, M, where perpendiculars drawn from it meet the lines given by pofition, and from X let parallels be drawn to thefe lines, meeting the perpendiculars from any point E in f, g, h.

Since thefe parallels Xf, Xg, Xh, interfect one another in the point X, fo as to make all the angles round it equal, they will divide the circumference of the circle which paffes through X and E, into three equal arches fg, gh, hf, (Lemma 2.) Therefore N, the centre of the circle, is the centre of gravity of the three points f, g, h, and the line YZ, paffing through N, and meeting the circumference, will be a diameter of the circle, and therefore YXZ is a right angle.

The fame demonftrated by Mr. Swale, Fig. 162, Pl. 11.

Let there be any number AB, CD, EF, GH, IK, LM, NO, &c. of right lines given by pofition, that are neither all parallel, nor interfecting each other in one point; two right lines WX, YZ may be found, that will be given by pofition, fuch, that if from any point P there be drawn perpendiculars PA, PC, PE, PG, PI, PL, PN, &c. to all the right lines AB, CD, EF, GH, IK, LM, NO, &c. given by pofition, and likewife PX, PZ, perpendiculars, to WX, YZ, the two right lines found, twice the fum of the fquares of the perpendiculars PA, PC, PE, PG, PI, PL, PN, &c. drawn to the right lines AB, CD, EF, GH, IK, LM, NO, &c. given by pofition, will be equal to the multiple of the fum

of

of the ſquares of the perpendiculars PX, PZ, drawn to WX, YZ, the two right lines found by the number of the right lines AB, CD, EF, GH, IK, LM, NO, &c. given by poſition, together with a given ſpace.

Suppoſe *ſeven* lines given by poſition: from any point P demit the perpendiculars PA, PC, PE, PG, PI, PL, PN; join AC, EC, GC, IC, LC, NC; take AQ, ER, GS, IT, LU, NV, equal to one-ſeventh of AC, EC, GC, IC, LC, NC, reſpectively; join PQ, PR, PS, PT, PU, PV; make Qn, Sa, perpendicular to PQ, PS, and equal to PR, PT, reſpectively; join Pn, Pa, perpendicular to which, and equal to PV, PU, reſpectively, take nc, ad; join Pc, Pd; make cW, dY perpendicular to Pc, Pd, to meet Pn, Pa, produced in W, Y; take Pm=one-third of Pn, Pu=one-third of Pa, at the points m, u, erect perpendiculars to meet ſemicircles deſcribed upon PW, PY, in X, Z: then the lines WX, YZ, joining the given points W, X; Y, Z; will be thoſe required.

For PX, PZ, being joined, will be perpendicular to WX, YZ.

Then, ſix times the ſquare of PA, together with the ſquare of PC, is equal to ſeven times the ſquare of PQ, together with the rectangle ACQ; ſix times the ſquare of PE, together with the ſquare of PC, is equal to ſeven times the ſquare of PR, together with the rectangle ECR; ſix times the ſquare of PG, together with the ſquare of PC, is equal to ſeven times the ſquare of PS, together with the rectangle GCS; ſix times the ſquare of PI, together with the ſquare of PC, is equal to ſeven times the ſquare of PT, together with the rectangle ICT; ſix times the ſquare of PL, together with the ſquare of PC, is equal to ſeven times the ſquare of PU, together with the rectangle LCU; and ſix times the ſquare of PN, together with the ſquare of PC, is

is equal to ſeven times the ſquare of PV, together with the rectangle NCV.

Therefore, ſix times the ſum of the ſquares of PA, PC, PE, PG, PI, PL, PN, is equal to ſeven times the ſum of the ſquares of PQ, PR, PS, PT, PU, PV, together with the ſum of the rectangles ACQ, ECR, GCS, ICT, LCU, NCV. But, ſeven times the ſum of the ſquares of PQ, PR, PV, is equal to ſeven times the ſquare of Pc, that is, equal to twenty-one times the ſquare of PX; ſeven times the ſum of the ſquares of PS, PT, PU, is equal to ſeven times the ſquare of Pd, that is, equal to twenty-one times the ſquare of PZ, and the ſum of the rectangles ACQ, ECR, GCS, ICT, LCU, NCV, is equal to forty-two times the ſum of the ſquares of AQ, ER, GS, IT, LU, NV. Therefore, ſix times the ſum of the ſquares of PA, PC, PE, PG, PI, PL, PN, is equal to twenty-one times the ſum of the ſquares of PX, PZ, together with forty-two times the ſum of the ſquares of AQ, ER, GS, IT, LU, NV; and therefore, twice the ſum of the ſquares of PA, PC, PE, PG, PI, PL, PN, is equal to ſeven times the ſum of the ſquares of PX, PZ, together with fourteen times the ſum of the ſquares of AQ, ER, GS, IT, LU, PN, that is, equal to the multiple of the ſum of the ſquares of PX, PZ, by the number of the lines given by poſition, together with a given ſpace.

Note, The foregoing method may be extended to any number of lines with the greateſt facility: for, if *nine* lines had been taken in the aſſumption, then AQ ſhould have been taken equal to one-ninth of AC, ER equal to one-ninth of EC, &c. alſo Pm equal to one-fourth of Pn, and Pu equal to one-fourth of Pa, and ſo on: the other parts of the conſtruction being preciſely the ſame.

PROP. XVII. THEO. XIV. *Fig.* 164, 165, 166, *Plate.* 11.

Demonstrated by Mr. Lowry.

Case 1. When the lines given by position are parallel to each other, fig. 164.

Let there be any number greater than three of right lines AL, BM, CN, DO, &c. given by position, and parallel to each other; three right lines XQ, YR, ZS, may be found, that will be given by position, such, that if from any point P there be drawn perpendiculars PA, PB, PC, PD, &c. to all the right lines given by position, and likewise the perpendiculars PX, PY, PZ, to the three lines found, thrice the sum of the squares of the perpendiculars PA, PB, PC, PD, &c. drawn to the right lines given by position, will be equal to the multiple of the sum of the squares of the perpendiculars PX, PY, PZ, drawn to the three lines found by the number of the lines given by position.

From P draw PABCD, &c. perpendicular to the right lines given by position, and by Prop. IX. find a point W for the points A, B, C, D, &c. In the right line PD take any point X, and divide XW in a, so that twice the multiple of the rectangle XaW by the number of the lines given by position, may be equal to the difference between twice the multiple of the square of XW by the number of the lines given by position, and thrice the sum of the squares of AW, BW, CW, DW, &c. Make WY equal to WA, and WZ equal to a X, and through the three points X, Y, Z, draw three right lines XQ, YR, ZS, parallel to the right lines given by position, and they will be three such lines as are required.

Because twice the multiple of the rectangle XaW, that is, the rectangle ZWY, by the number

of

of the lines given by poſition, is equal to the difference between twice the multiple of the ſquare of XW by the ſame number, and thrice the ſum of the ſquares of AW, BW, CW, DW, &c. and ſince XW is equal to the ſum of WY, WZ, the ſquare of XW is equal to the ſum of the ſquares of WY, WZ, together with twice the rectangle YWZ, and therefore the multiple of the ſum of the ſquares of XW, YW, ZW, by the number of the lines given by poſition, is equal to thrice the ſum of the ſquares of AW, BW, CW, DW, &c. But (Prop. IX.) thrice the ſum of the ſquares of AP, BP, CP, DP, &c. is equal to thrice the ſum of the ſquares of AW, BW, CW, DW, &c. together with thrice the multiple of the ſquare of PW, by the number of the lines given by poſition; therefore, thrice the ſum of the ſquares of AP, BP, CP, DP, &c. is equal to the multiple of the ſum of the ſquares of XW, YW, ZW, by the number of the lines given by poſition, together with thrice the multiple of the ſquare of PW by the ſame number.

Again, becauſe XW is equal to the ſum of WY, WZ, it follows, from Prop. IX. that the ſum of the ſquares of PX, PY, PZ, is equal to the ſum of the ſquares of XW, YW, ZW, together with the multiple of the ſquare of PW, by the number of the lines given by poſition; therefore thrice the ſum of the ſquares of AP, BP, CP, DP, &c. is equal to the multiple of the ſum of the ſquares of PX, PY, PZ, by the number of the lines given by poſition.

Caſe 2. When the lines given by poſition interſect each other in a point, fig. 165.

Let there be any number greater than three of right lines AB, AC, AD, AE, &c. given by poſition, and interſecting each other in the point A; three right lines AX, AY, AZ, may be found, that will be given by poſition, ſuch, that if from

any point P there be drawn perpendiculars PB, PC, PD, PE, &c. to all the right lines given by poſition, and likewiſe the perpendiculars PX, PY, PZ, to the three lines found, thrice the ſum of the ſquares of the perpendiculars PB, PC, PD, PE, &c. drawn to the right lines given by poſition, will be equal to the multiple of the ſum of the ſquares of the perpendiculars PX, PY, PZ, drawn to the three lines found by the number of the lines given by poſition.

Draw AP, and upon it, as a diameter, let a circle be deſcribed, interſecting the lines given by poſition in B, C, D, E, &c. By Prop. IX. find a point R for the points B, C, D, E, &c. and by Prop. XV. find two right lines AI, AK, for the right lines AB, AC, AD, AE, &c. given by poſition, and the point P. From any point Z, in the circumference of the circle, draw ZR, and continue it, ſo that RZ may be double of RS. Let O be the centre of the circle, and join OS, and let XSY be drawn perpendicular to SO; join AX, AY, AZ, and they will be three ſuch lines as are required.

PB, PC, PD, PE, &c. PI, PK, PX, PY, and PZ, being joined, will be perpendicular to AB, AC, AD, AE, &c. AI, AK, AX, AY, and AZ; let PS, PR, and OZ be joined.

Then, Prop. XV. twice the ſum of the ſquares of PB, PC, PD, PE, &c. is equal to the multiple of the ſum of the ſquares of PI, PK, by the number of the lines given by poſition, therefore thrice the ſum of the ſquares of PB, PC, PD, PE, &c. is equal to thrice the multiple of the ſum of the ſquares of PI, PK, by half the number of the lines given by poſition. Again, ſince XY is perpendicular to SO, XY will be biſected in S, therefore the ſum of the ſquares of PX, PY, is equal to twice the ſum of the ſquares of SY, SP;

and,

and, Prop. II. the ſquare of PZ, together with twice the ſquare of SP, is equal to ſix times the ſquare of RS, together with thrice the ſquare of RP; therefore the ſum of the ſquares of PX, PY, PZ, is equal to twice the ſquare of SY, together with ſix times the ſquare of RS, together with thrice the ſquare of RP. But the ſquare of SY is equal to the difference of the ſquares of OP, OS, and (Prop. II.) the ſquare of OP, together with twice the ſquare of OS, is equal to ſix times the ſquare of RS, together with thrice the ſquare of OR, and the ſquare of RI is equal to the difference of the ſquares of OP, OR; therefore, ſix times the ſquare of RS, together with twice the ſquare of SY, is equal to thrice the ſquare of RI, and therefore, the ſum of the ſquares of PX, PY, PZ, is equal to thrice the ſum of the ſquares of PR, IR. But twice the ſum of the ſquares of PR, IR, is equal to the ſum of the ſquares of PI, PK, therefore the multiple of the ſum of the ſquares of PX, PY, PZ, by the number of the lines given by poſition, is equal to thrice the multiple of the ſum of the ſquares of PI, PK, by half the number of the lines given by poſition. But it has been ſhewn, that thrice the ſum of the ſquares of PB, PC, PD, PE, &c. is equal to thrice the multiple of the ſum of the ſquares of PI, PK, by half the number of the lines given by poſition, therefore thrice the ſum of the ſquares of PB, PC, PD, PE, &c. is equal to the multiple of the ſum of the ſquares of PX, PY, PZ, by the number of the lines given by poſition.

Caſe 3. When the lines given by poſition are neither all parallel, nor interſecting each other in one point, fig. 166.

Let there be any number greater than three of right lines AB, BC, CD, AD, &c. given by poſition, that are neither all parallel, nor interſecting each

other in one point, three right lines Ym, Zn, Kr, may be found, that will be given by poſition, ſuch, that if from any point X there be drawn perpendiculars XE, XF, XG, XH, &c. to all the right lines given by poſition, and likewiſe perpendiculars Xm, Xn, Xr, to the three lines found, thrice the ſum of the ſquares of the perpendiculars XE, XF, XG, XH, &c. drawn to the right lines given by poſition, will be equal to the multiple of the ſum of the ſquares of the perpendiculars Xm, Xn, Xr, drawn to the three lines found by the number of the lines given by poſition.

By Prop. XVI. find two right lines PV, PW, for the point X, and the right lines AB, BC, CD, DA, &c. given by poſition. From P draw PM, PO, PR, PQ, &c. perpendicular, and PL, PN, PS, PU, &c. parallel to AB, BC, CD, DA, &c. and let PL, PN, PS, PU, &c. meet the perpendiculars XE, XF, XG, XH, &c. drawn from the point X to the right lines AB, BC, CD, DA, &c. in L, N, S, U, &c. Find, by the preceding caſe, three right lines Pa, Pb, Pc, for the right lines PL, PN, PS, PU, &c. given by poſition, and the point X, and from any point, as c, in Pc, draw cq parallel to Pa, and let it meet Pb in q.

Then, by Prop. XVI. find the interſection w of two right lines, for the right lines Pq, Pc, and cq, and the point X, From w draw wf, wg, wh, perpendicular to cq, Pc, Pq. Let a ſquare be found which ſhall have to the ſum of the ſquares of PM, PO, PR, PQ, &c. the ſame ratio that three has to the number of the right lines given by poſition. Divide this ſquare into three others, whoſe ſides ſhall have the ſame ratio to each other as the three lines wf, wg, wh. Draw PK, PY, PZ, parallel to wf, wg, wh, and thereon take PK, PY, PZ, equal to the ſides of the three ſquares juſt found, that is, on lines which are parallel to thoſe to which the

ſides

ſides of the ſquares are reſpectively proportional. Then draw Ym, Zn, Kr, perpendicular to PY, PZ, PK, and they will be three ſuch lines as are required.

Draw Xm, Xn, Xr, Xa, Xb, Xc, XV, XW reſpectively parallel to Ym, Zn, Kr, Pa, Pb, Pc, PV, PW. Then (Prop. XVI.) twice the ſum of the ſquares of XE, XF, XG, XH, &c. is equal to the multiple of the ſum of the ſquares of XV, XW, by the number of the lines given by poſition, together with twice the ſum of the ſquares of PO, PM, PR, PQ, &c. and (becauſe of the parallels) twice the ſum of the ſquares of XL, XN, XS, XU, &c. is equal to the multiple of the ſum of the ſquares of XV, XW, by the number of the lines given by poſition, therefore twice the ſum of the ſquares of XE, XF, XG, XH, &c. is equal to twice the ſum of the ſquares of PM, PO, PR, PQ, &c. together with twice the ſum of the ſquares of XL, XN, XS, XU, &c. therefore thrice the ſum of the ſquares of XE, XF, XG, XH, &c. is equal to thrice the ſum of the ſquares of PM, PO, PR, PQ, &c. together with thrice the ſum of the ſquares of XL, XN, XS, XU, &c. In the ſame way it is ſhewn, that the ſum of the ſquares of Xm, Xn, Xr, is equal to the ſum of the ſquares of Xa, Xb, Xc, together with the ſum of the ſquares of PY, PZ, PK, therefore the multiple of the ſum of the ſquares of Xm, Xn, Xr, by the number of the lines given by poſition, is equal to the multiple of the ſum of the ſquares of Xa, Xb, Xc, by the ſame number, together with the multiple of the sum of the ſquares of PY, PZ, PK, by the ſame number.

Again, by the preceding caſe, thrice the ſum of the ſquares of XL, XN, XS, XU, &c. is equal to the multiple of the ſum of the ſquares of Xa, Xb, Xc, by the number of the lines given by poſition, and by conſtruction, thrice the ſum of the

ſquares

ſquares of PM, PO, PR, PQ, &c. is equal to the multiple of the ſum of the ſquares of PY, PZ, PK, by the number of the lines given by poſition; therefore, thrice the ſum of the ſquares of PM, PO, PR, PQ, &c. together with thrice the ſum of the ſquares of XL, XN, XS, XU, &c. is equal to the multiple of the ſum of the ſquares of XA, Xb, Xc, by the number of the lines given by poſition, together with the multiple of the ſum of the ſquares of PY, PZ, PK, by the ſame number; therefore thrice the ſum of the ſquares of XE, XF, XG, XH, &c. is equal to the multiple of the ſum of the ſquares of Xm, Xn, Xr, by the number of the lines given by poſition.

The ſame demonſtrated by Mr. Swale, Fig. 172, Pl. 12.

Let there be any number greater than three of right lines Aa, Bb, Cc, Dd, Ee, Ff, Gg, &c. given by poſition; three right lines OQ, RS, TV, may be found, that will be given by poſition, ſuch, that if from any point P there be drawn perpendiculars PA, PB, PC, PD, PE, PF, PG, &c. to all the right lines Aa, Bb, Cc, Dd, Ee, Ff, Gg, &c. given by poſition, and likewiſe PQ, PS, PV, perpendiculars to OQ, RS, TV, the three right lines found, three times the ſum of the ſquares of the perpendiculars PA, PB, PC, PD, PE, PF, PG, &c. drawn to the right lines Aa, Bb, Cc, Dd, Ee, Ff, Gg, &c. given by poſition, will be equal to the multiple of the ſum of the ſquares of the perpendiculars PS, PQ, PV, drawn to OQ, RS, TV, the three right lines found by the number of the right lines Aa, Bb, Cc, Dd, Ee, Ff, Gg, &c. given by poſition.

Let there be *ſeven* lines given by poſition.

From any point P demit the perpendiculars PA, PB, PC, PD, PE, PF, PG; join AB, CB, DB, EB, FB, GB; take AH, CI, DK, EL, FM, GN,

GN, equal to one-ſeventh of AB, CB, DB, EB, FB, GB reſpectively; join PH, PI, PK, PL, PM, PN; join NH, IK, LM, which biſect in h, i, l reſpectively, and join Ph, Pi, Pl: perpendicular to Ph, Pi, Pl, and equal to hH (hN), iK, (iI), lM (lL) reſpectively, take hm, ip, lv; join Pm, Pp, Pv: perpendicular to GB, DB, FB, and equal AH, CI, EL reſpectively, take Ns, Kw, Mx, and join Gs, Dw, Fx; then perpendicular to Pm, Pp, Pv, take mn, pq, vt, equal to Gs, Dw, Fx; join Pn, Pq, Pt, and demit, on Pm, Pp, Pv, the ⊥'s no, qr, tu; make mO, pR, vT equal thrice mo, pr, vu reſpectively; produce the perpendiculars mn, pq, vt, to meet ſemicircles deſcribed upon PO, PR, PT, in Q, S, V; then through the given points O, Q; R, S; T, V; drawing OQ, RS, TV, they will be the three lines that were to be found.

For PQ, PS, PV, being joined, will be perpendicular to QO, SR, VT.

Since ſix times the ſquare of PA, together with the ſquare of PB, is equal to ſeven times the ſquare of PH, together with the rectangle ABH; ſix times the ſquare of CP, together with the ſquare of PB, is equal to ſeven times the ſquare of PI, together with the rectangle CBI; ſix times the ſquare of PD, together with the ſquare of PB, is equal to ſeven times the ſquare of PK, together with the rectangle DBK; ſix times the ſquare of PE, together with the ſquare of PB, is equal to ſeven times the ſquare of PL, together with the rectangle EBL; ſix times the ſquare of PF, together with the ſquare of PB, is equal to ſeven times the ſquare of PM, together with the rectangle FBM; and ſix times the ſquare of PG, together with the ſquare of PB, is equal to ſeven times the ſquare of PN, together with the rectangle GBN: ſix times the ſum of the ſquares of PA, PB, PC, PD, PE, PF, PG, will be equal to ſeven times the ſum of the ſquares of PH, PI,

PI, PK, PL, PM, PN, together with the sum of the rectangles ABH, CBI, DBK, EBL, FBM, GBN. But seven times the sum of the squares of PH, PN, is equal to fourteen times the sum of the squares of Ph, hH, that is, equal to fourteen times the square of Pm; seven times the sum of the squares of PI, PK, is equal to fourteen times the sum of the squares of Pi, iK, that is, equal to fourteen times the square of Pp; and seven times the sum of the squares of PL, PM, is equal to fourteen times the sum of the squares of Pl, lM, that is, equal to fourteen times the square of Pv, and the sum of the rectangles GBN, ABH, is equal to forty-two times the sum of the squares of GN, AH, that is, equal to forty-two times the square of GS (mn), that is, equal to fourteen times the square of mQ; the sum of the rectangles CBI, DBK, is equal to forty-two times the square of Dw (pq), that is, equal to fourteen times the square of pS; and the sum of the rectangles EBM, FBL, is equal to forty-two times the square of Fx (vt), that is, equal to fourteen times the square of vV: therefore six times the sum of the squares of PA, PB, PC, PD, PE, PF, PG, is equal to fourteen times the sum of the squares of Pm, mQ, Pp, pS, Pv, vV, that is, equal to fourteen times the sum of the squares of PQ, PS, PV; and therefore three times the sum of the squares of PA, PB, PC, PD, PE, PF, PG, is equal to seven times the sum of the squares of PQ, PS, PV, that is, equal to the multiple of the sum of the squares of the perpendiculars PQ, PS, PV, by the number of the lines given by position.

Note, The above method may be easily extended to any number of lines, by pursuing the directions in my Demon. to Prop. XVI.

The

The same demonstrated by Dr. Small, Fig. 170, 171, 173, Pl. 12, *and* Fig. 180, Pl. 13.

'Let any number, *m*, greater than three of straight lines be given by position, three straight lines may be found, which will be given by position, such, that if from any point there be drawn perpendiculars to the lines given by position, and to the three lines found, thrice the sum of the squares of the perpendiculars to the lines given by position will be equal to the sum of the squares of the perpendiculars drawn to the three lines found, multiplied by the number *m*.

Let m be $=4$.

Case 1. When the lines (fig. 173.) AF, BG, CH, DK, given by position, are all parallel. Let a perpendicular from any point E meet the parallels in the points A, B, C, D, and let L be the centre of gravity of these points. Assume in AL any point X, and let Y and Z, on the opposite side of L, be such, that $LY+LZ=LX$, and also $LX^2+LY^2+LZ^2=\frac{3}{4}(LA^2+LB^2+LC^2+LD^2)$; then if the assumed point X be given, the points Y and Z will also be given. Draw through the points X, Y, Z, straight lines parallel to AF, and they will be the lines required.

It is plain that L is the centre of gravity of the points X, Y, Z; and because it is also the centre of gravity of the points A, B, C, D, $3(EA^2+EB^2+EC^2+ED^2)=3(LA^2+LB^2+LC^2+LD^2)+3\cdot4\cdot EL^2$, (Theor. 6.); and for the same reason $4(EX^2+EY^2+EZ^2)=4(LX^2+LY^2+LZ^2)+4\cdot3\cdot EL^2$. But by construction $3(LA^2+LB^2+LC^2+LD^2)=4(LX^2+LY^2+LZ^2)$. Therefore $3(EA^2+EB^2+EC^2+ED^2)=4(EX^2+EY^2+EZ^2)$.

Case 2. When the lines (fig. 180.) AB, AC, AD, AE, given by position, intersect one another in the same point A.

Let

Let G be the centre of gravity of the four points B, C, D, E, in the circumference of the circle of which AF is the diameter, (Theor, 6.), and let AH, AK, be two lines, whose position is given, such, that $2(FB^2+FC^2+FD^2+FE^2)=4(FH^2+FK^2)$, (Theo. 12.) From any point X in the circumference draw, through G, the line XGL, so that $XG=2GL$; and through L draw YLZ, to meet the circumference in Y, Z, and so as to be bisected in L. Join AX, AY, AZ, and FX, FY, FZ.

$3(FB^2+FC^2+FD^2+FE^2) = 6(FH^2+FK^2)$, (Theor. 12.), and $4(FX^2+FY^2+FZ^2)=6(FH^2+FK^2)=3(FB^2+FC^2+FD^2+FE^2)$. Therefore AX, AY, AZ, are the three lines required to be found.

Case 3. When the lines (fig. 170.) AB, BC, CD, DA, are not parallel, and do not intersect one another in the same point.

Let X be a point so related to the lines AB, BC, CD, DA, that it shall be the centre of gravity of the four points L, M, N, O, where they are intersected by the perpendiculars XL, XM, XN, XO, drawn to them from X, (Theor. 13.); and let XP, XQ, XR, XS, be drawn from X parallel to AB, BC, CD, DA, and let them meet the perpendiculars to these lines, from E, in P, Q, R, S. Let Xa, Xb, Xc, be three straight lines, such, that $3(EP^2+EQ^2+ER^2+ES^2)=4(Ea^2+Eb^2+Ec^2)$, (case 2. of this). Describe a triangle def, (fig. 171.) having the angle def=aXb, and the angle dfe=bXc. Let g be a point in that triangle, such, as to be the centre of gravity of the three points h, k, l, where perpendiculars drawn from it meet the sides, (Theor. 13.).

Describe a square $=\frac{3}{4}(XL^2+XM^2+XN^2+XO^2)$, and divide it into three squares whose sides Xm, Xn, Xo, shall have the mutual ratios of gh, gk, gl. Through X draw Xm, Xn, Xo, perpendicular

Xa, Xb, Xc, and through m, n, o, draw , qp, perpendicular to Xm, Xn, Xo, and Ea, Eb, Ec, in x, y, z. We have, by 13.

$G^2+EH^2+EK^2) = 3(EP^2+EQ^2+ER^2+ES^2) + 3$ $XM^2+XN^2+XO^2)$, and alſo

$y^2+Ez^2) = 4(Ea^2+Eb^2+Ec^2)+4(Xm^2+Xn^2+Xo^2)$.

conſtruction

$Q^2+ER^2+ES^2) = 4(Ea^2+Eb^2+Ec^2)$, and by Caſe 2.

$3(XL^2+XM^2+XN^2+XO^2) = 4(Xm^2+Xn^2+Xo^2)$.

re

$-EG^2+EH^2+EK^2) = 4(Ex^2+Ey^2+Ez^2)$.

re mp, nq, qp, are the lines required to be

three lines found in this Theorem are de-d, in their poſition, only relatively to one , and not abſolutely; becauſe, in the con-n of each of the caſes, an arbitrary ſup-is unavoidably introduced, and of conſe-there are innumerable ſets of lines, within limits however, that all equally anſwer the ons required in the Propoſition. When one is aſſumed as given in poſition, the other neceſſarily determined.

P. XVIII. THEO. XV. Fig. 99, Pl. 7.

Demonſtrated by Mr. Lowry.

there be any number of right lines AL, BM, c. given by poſition, and parallel to each and let *a*, *b*, *c*, &c. be given magnitudes, as n number as there are right lines given by n; a right line XY may be found parallel to ht lines given by poſition, ſuch, that if from oint P there be drawn perpendiculars PA, C, &c. to the right lines given by poſition, ewiſe a perpendicular PX to the right line

K k found,

found, the square of PA, together with the space to which the square of PB has the same ratio that *a* has to *b*, together with the space to which the square of PC has the same ratio that *a* has to *c*, and so on, will be equal to the space to which the square of PX has the same ratio that *a* has to the sum of *a*, *b*, *c*, &c. together with a given space.

From P draw PAB, &c. perpendicular to AL, BM, CN, &c. Find the point X by Prop. X. for the given points A, B, C, &c. and the given magnitudes *a*, *b*, *c*, &c. Through X let XY be drawn parallel to the right lines given by position, and it will be the line required.

For, (Prop. X.) the square of PA, together with the space to which the square of PB has the same ratio that *a* has to *b*, together with the space to which the square of PC has the same ratio that *a* has to *c*, and so on, is equal to the square of AX, together with the space to which the square of BX has the same ratio that *a* has to *b*, together with the space to which the square of CX has the same ratio that *a* has to *c*, and so on, together with the space to which the square of PX has the same ratio that *a* has to the sum of *a*, *b*, *c*, &c. But the square of AX, together with the space to which the square of BX has the same ratio that *a* has to *b*, together with the space to which the square of CX has the same ratio that *a* has to *c*, and so on, is equal to a given space; therefore the square of PA, together with the space to which the square of PB has the same ratio that *a* has to *b*, together with the space to which the square of PC has the same ratio that *a* has to *c*, and so on, is equal to the space to which the square of PX has the same ratio that *a* has to the sum of *a*, *b*, *c*, &c. together with a given space.

The same demonstrated by Mr. Swale, Fig. 167, Pl. 11.

Let there be any number of right lines AG, HL, BC, &c. given by position, and parallel to each other; and let *a*, *b*, *c*, &c. be given magnitudes, as many in number as there are right lines given by position; a right line DF may be found parallel to the right lines given by position, such, that if from any point P there be drawn a common perpendicular AB to the right lines given by position, and to the right line found; the square of PA, together with the space to which the square of PB has the same ratio that *a* has to *b*, together with the space to which the square of PH has the same ratio that *a* has to *c*, and so on, will be equal to the space to which the square of PD has the same ratio that *a* has to the sum of *a*, *b*, *c*, &c. together with a given space.

Let there be two right lines AG, BC, given by position, and parallel to each other. Join A, B, by the perpendicular AB; make BD to DA in the ratio of *a* to *b*; draw DF parallel to BC, and it will be the line required. Make BC equal to BA; join AC, and let it meet DF in F; from any point P in AB draw PE parallel to BC; join BF, DE, DI (I being the point of intersection of PE, BF).

Because AB is equal to BC, AP will be equal to PE, and AD equal to DF, and the square of PA will be equal to the square of PE, that is, equal to twice the triangle of APE. Again, the square of BP is to the rectangle BPI as BP to PI, that is, as BD to DF, that is, as *a* to *b*; therefore the rectangle BPI, that is, twice the triangle BPI is the space to which the square of BP has the same ratio that *a* has to *b*, and therefore, the square of PA, together with the space to which the square of BP has the same ratio that *a* has to *b*, is equal to twice the

triangle APE, together with twice the triangle BPI, that is, equal to twice the triangle AFB, together with twice the triangle IFE. Moreover, DP is to DB as FE to FC, that is, as FI to FB, that is, as EI to CB; therefore the ſquare of PD is to the rectangle contained by DP, EI, as DB, to BC, that is, as *a* to the ſum of *a*, *b*; but the rectangle contained by DP, EI, is equal to twice the triangle IDE, that is, equal to twice the triangle IFE, therefore twice the triangle IFE is the ſpace to which the ſquare of PD has the ſame ratio that *a* has to the ſum *a*, *b*; and therefore the ſquare of PA, together with the ſpace to which the ſquare of BP has the ſame ratio that *a* has to *b*, is equal to the ſpace to which the ſquare of PD has the ſame ratio that *a* has to the ſum of *a*, *b*, together with twice the triangle AFB, that is, equal to the ſpace to which the ſquare of PD has the ſame ratio that *a* has to the ſum of *a*, *b*, together with a given ſpace.

Dr. Small ſays, that this Propoſition is related to the 13th, juſt as the 10th Propoſition is to the 9th.

Cor. Let there be any number of right lines given by poſition, and parallel to each other; a right line may be found, that will be given by poſition, ſuch, that if from any point there be drawn right lines in given angles to the right lines given by poſition, and likewiſe there be drawn a perpendicular to the right line found, the ſum of the ſquares of the lines drawn in given angles to the right lines given by poſition, will be equal to the ſpace to which the ſquare of the perpendicular drawn to the line found has a given ratio, together with a given ſpace.

ARTICLE

ARTICLE XLVIII.

Important Corrections, by the late ingenious and learned JOHN LANDEN, ESQ. F.R.S. *taken from the London Magazine Improved, for the Year* 1785.

PROBLEM.

MR. SIMPSON, p. 38 of his *Dissertations*, proposes to determine the height of the tides at any planet: it is here proposed to examine whether his computation be true or false; and if false, to point out the error.

SOLUTION. Fig. 168, Plate 12.

The force of a particle at Q, urging it from AOE, in a direction parallel to a line joining the centres O and C of the two bodies, is $2fmx \div d^3$, &c. which, in the direction of the tangent QF, is nearly $=(2fm \div d^3) \cdot$RD, as computed by Mr. Simpson; QR, RD, being perpendicular to OC, OQ; and OC being $=d$, OP$=1$, OR$=x$. But besides that *force*, there is another, in the direction QR, $=fm\sqrt{1-x^2} \div d^3$ which that gentleman has not considered; and from this last-mentioned force arises an additional one $=(fm \div d^3) \cdot$RD, in the direction QF. Therefore, instead of $\frac{1}{3} : \frac{-2B}{3 \cdot 5} \times$RD$:: f : \frac{2fm}{d^3} \times$ RD, we have $\frac{1}{3} : (-2B \div 3 \cdot 5) \cdot$RD$:: f : (3fm \div d^3) \cdot$RD, and consequently B$= - 15m \div 2d^3$. Hence, our author having shewn that OP^2 will be to OA^2 as 1 to $1+$B, we find OP—OA nearly$=(15m \div 4d^3) \cdot$ OP: and thus the tides at the body O, by the action of

of the body C, appear to be greater in the proportion of 3 to 2 than his computation makes them. The body O is taken as a perfect sphere, except by so much as it differs therefrom through the cause under consideration (which will cause no sensible error in the solution); and the quantity of matter in that body O, to the quantity of matter in the body C, is supposed as 1 to m. The accelerative force of the body O on a particle at Q, in the direction QO, is denoted by f.

The force, which Mr. Simpson has omitted, is derived (by resolution) from that of the body C in the direction QC.

PROBLEM.

Mr. Emerson, at p. 421, of the *second edition of his Fluxions*, has computed the height of the tides. Is his computation right, or wrong? If wrong, please to shew how it may be rectified.

SOLUTION. Fig. 169, Plate 12.

Mr. Emerson (to whose characters I refer) makes the gravity at P the same as at A; which, though the difference is very small, occasions a very considerable error in the conclusion. His value of the perturbating force of S, on a particle at D, is also erroneous (being $2fy \div 3$ instead of fy); and he has omitted the force of S on a particle at E, in the direction EC.

Let a^2 be to b^2 as 1 to $1+B$; then will $a-b$ be $= -a^2B \div (a+b) = -aB \div 2$ nearly; and the gravity at A will be to the gravity at P as b to $(1+2B \div 5) \cdot a$. Therefore, instead of his equation $gy\dot{y} \div a - fy\dot{y} = gx\dot{x} \div b$, we have the *whole fluents* of $gy\dot{y} \div a - \frac{2}{3}fy\dot{y} =$ the *whole fluents* of $(1+\frac{2}{5}B) \cdot agx\dot{x} \div b^2$

$\div b^2+4R\pi^2 x\dot{x}\div p^2\sqrt{R^2+x^2}$: whence, by taking the fluents, we have $ga-fa^2=(1+\frac{2}{5}B)\cdot ag$; and confequently B being by that equation $=-5af\div 2g$, $a-b$ (inftead of being $=a^2f\div g$) will be $=5a^2f\div 4g$ *. Which agrees with Mr. Mac Laurin's computation, and with my correction of Mr. Simpfon's given above: a being $=$ AC; $b=$ CP; R$=$CS; $p=$ the periodical time of the earth round the fun in feconds; $g=32\cdot 2$ feet; $\pi=3\cdot 1416$; $f=12\,\pi^2\div p^2$; $x=$ CE; $y=$ CD; and the *whole fluent* of $4R\pi^2 x\dot{x}\div p^2\sqrt{R^2+x^2}=2\pi^2b^2\div p^2=fb^2\div 6=fa^2\div 6$ nearly.

* From this equation Mr. R. Simpfon derived the Theorem ufed in his folution to the 43d queftion of this work.

ARTICLE XLIX.

Landen's *Theorems, for the Calculation of Fluents.*

TABLE IV.

(Continued from page 313.)

THEOREM XV.

$$\dot{F}=x^{-\frac{2}{3}}\dot{x}\div(a^2+x^2)^{\frac{1}{2}}=\tfrac{3}{2}y^{-\frac{1}{2}}\dot{y}\div(b^3+y^3)^{\frac{1}{2}}.$$

$$F=K-\tfrac{3}{2}a^{-\frac{3}{2}}D.$$

Here $z=(a^{\frac{2}{3}}+x^{\frac{2}{3}})\div x^{\frac{2}{3}}=(b+y)\div y.$

THEOREM

THEOREM XVI.

The *wh.fl.*of $x^{-\frac{2}{3}}\dot{x}\div(a^2+x^2)^{\frac{1}{2}}$ is $=2\cdot3^{\frac{1}{4}}a^{-\frac{2}{3}}\mathrm{P}\div(3^{\frac{1}{2}}+1)$.

THEOREM XVII.

$\dot{\mathrm{F}}=x^{-\frac{1}{3}}\dot{x}\div(a^2+x^2)^{\frac{1}{2}}=\frac{3}{2}\dot{y}\div(b^3+y^3)^{\frac{1}{2}}$.

$\mathrm{F}=\mathrm{K}+\frac{3}{2}a^{-\frac{1}{3}}\mathrm{D}$.

Here $z=(a^{\frac{2}{3}}+x^{\frac{2}{3}})\div a^{\frac{2}{3}}=(b+y)\div b$.

THEOREM XVIII.

The *wh.fl.*of $x^{-\frac{1}{3}}\dot{x}\div(a^2+x^2)^{\frac{1}{2}}$ is $=2\cdot3^{\frac{1}{4}}a^{-\frac{1}{3}}\mathrm{P}\div(3^{\frac{1}{2}}+1)$.

THEOREM XIX.

$\dot{\mathrm{F}}=x^{\frac{1}{3}}\dot{x}=(a^2+x^2)^{\frac{1}{2}}=\frac{3}{2}y\dot{y}\div(b^3+y^3)^{\frac{1}{2}}$

$\mathrm{F}=\mathrm{K}+\frac{3}{2}a^{\frac{1}{3}}\times(\mathrm{C}-\mathrm{D})$.

Here $z=(a^{\frac{2}{3}}+x^{\frac{2}{3}})\div a^{\frac{2}{3}}=(b+y)\div b$.

Note, The *whole* fluent is infinite.

THEOREM

THEOREM XX.

$\dot{F} = x^{-\frac{2}{3}}\dot{x} \div (a^2+x^2)^{\frac{1}{2}} = \frac{3}{2}y^{-\frac{3}{2}}\dot{y} \div (b^3+y^3)^{\frac{1}{2}}$

$F = K - \frac{3}{4}a^{\frac{2}{3}} \times (C-D) + \frac{3}{2}x^{-\frac{1}{3}}\sqrt{(a^2+x^2)}.$

Here $z = (a^{\frac{2}{3}}+x^{\frac{2}{3}}) \div x^{\frac{2}{3}} = (b+y) \div y.$

Note, The *whole* fluent is infinite.

THEOREM XXI.

$\dot{F} = x^{-\frac{2}{3}}\dot{x} \div (a^2-x^2)^{\frac{1}{3}} = \frac{3}{2}y^{-\frac{1}{2}}\dot{y} \div (b^3-y^3)^{\frac{1}{3}}$

$F = -$ fluent of $w^{-\frac{1}{3}}\dot{w} \div (4a^4+w^4)^{\frac{1}{2}}$, to be found by Theo. 17.

Here $w = (a^2-x^2) \div x = (b^3-y^3) \div y^{\frac{3}{2}}.$

THEOREM XXII.

The *wh.* fl. $x^{-\frac{2}{3}}\dot{x} \div (a^2-x^2)^{\frac{1}{3}}$ is $= 2^{\frac{2}{3}}3^{\frac{1}{4}}a^{-\frac{1}{3}}P \div (3^{\frac{1}{2}}+1).$

THEOREM XXIII.

$\dot{F} = \dot{x} \div (a^2-x^2)^{\frac{1}{3}} = \frac{3}{2}y^{\frac{1}{2}}\dot{y} \div (b^3-y^3)^{\frac{1}{3}}.$

$\dot{F} = -$ fl. $w^{\frac{1}{3}}\dot{w} \div (a^2-w^2)^{\frac{1}{2}}$, to be found by Theo. 5.

Here $w = \sqrt{(a^2-x^2)} = \sqrt{(b^3-y^3)}.$

THEOREM

THEOREM XXIV.

The *whole* fluent of $\dot{x} \div (a^2 - x^2)^{\frac{1}{3}}$ is $= 3^{\frac{1}{4}} a^{\frac{1}{3}} Q$.

THEOREM XXV.

$\dot{F} = x^{\frac{1}{3}} \dot{x} \div (a^2 - x^2)^{\frac{1}{3}} = \frac{3}{2} y \dot{y} \div (b^3 - y^3)^{\frac{1}{3}}$.

$F = -\frac{3}{4} w^{\frac{2}{3}} + \frac{1}{2} \text{fl.} w^{\frac{2}{3}} \dot{w} \div (4a^2 + w^2)^{\frac{1}{2}}$, to be found by Theo. 20.

Here $w = (a^2 - x^2) \div x = (b^3 - y^3) \div y^{\frac{3}{2}}$.

THEOREM XXVI.

The *wh.* fl. of $x^{\frac{1}{3}} \dot{x} \div (a^2 - x^2)^{\frac{1}{3}}$ is $= 3^{\frac{1}{4}} a^{\frac{2}{3}} Q \div 2^{\frac{1}{3}}$.

THEOREM XXVII.

$\dot{F} = x^{\frac{2}{3}} \dot{x} \div (a^2 - x^2)^{\frac{1}{3}} = \frac{3}{2} y^{\frac{3}{2}} \dot{y} \div (b^3 - y^3)^{\frac{1}{3}}$.

$\dot{F} = a^{\frac{1}{3}} w^{\frac{5}{3}} \div (a^2 + w^2)^{\frac{1}{2}} - \frac{2}{3} a^{\frac{1}{3}} \text{fl.} w^{\frac{2}{3}} \dot{w} \div (a^2 + w^2)^{\frac{1}{2}}$ Th. 20.

Here $w = ax \div (a^2 - x^2) = b^{\frac{3}{2}} y^{\frac{3}{2}} \div (b^3 - y^3)^{\frac{1}{3}}$.

THEOREM XXVIII.

The *wh.* fluent of $x^{\frac{2}{3}} \dot{x} \div (a^2 - x^2)^{\frac{1}{3}}$ is $= 2^{\frac{1}{3}} a Q \div 3^{\frac{3}{4}}$.

THEOREM

THEOREM XXIX.

$\dot{F} = x^{-\frac{2}{3}}\dot{x} \div (x^2 - a^2)^{\frac{1}{3}} = \frac{3}{2}y^{-\frac{1}{2}}\dot{y} \div (y^3 - b^3)^{\frac{1}{3}}$.

$F = \text{fl.}\, w^{-\frac{1}{3}}\dot{w} \div (4a^2 + w^2)^{\frac{1}{2}}$, to be found by Theo. 17.

Here $w = (x^2 - a^2) \div x = (y^3 - b^3) \div y^{\frac{3}{2}}$.

THEOREM XXX.

The *wh.* fl. $x^{-\frac{2}{3}}\dot{x} \div (x^2 - a^2)^{\frac{1}{3}}$ is $= 2^{\frac{2}{3}} 3^{\frac{1}{4}} a^{-\frac{1}{3}} P \div (3^{\frac{1}{2}} + 1)$;

(To be continued.)

ARTICLE L.

Atwood's *Investigations on Watch Balances.*

(Continued from page 317.)

SUPPOSE a watch to be adjusted to mean time when the semiarc of the balance's vibration is = BO, (fig. 81, pl. 5.) and let this semiarc be afterwards diminished to IO; the time shewn by the watch in any given portion of mean time t, when the semiarc of vibration is IO, will be equal

$t \times (BO \div IO)^{\frac{1-n}{2}}$; and if t is put $= 24^h$, the alteration of the daily rate, in consequence of the diminution of the semiarc of vibration from BO to IO, will be $24^h \times ((BO \div IO)^{\frac{1-n}{2}} - 1)$*. To apply this proposition, let a case be assumed; suppose a watch

* From this general expression it appears, that when $n = 1$, that is, when the spring's elastic force varies in the precise ratio of the angular distances of the balance from the quiescent position, the alteration of the daily rate, in consequence of a diminution of the arc of vibration. is $= 0$; because, in that case, $(BO \div IO)^{\frac{1-n}{2}} = 1$, and $(BO \div IO)^{\frac{1-n}{2}} - 1 = 0$. When n is less than 1, or when the force varies in a less ratio than that of the distances from quiescence, the rate will be accelerated, because in that case $(BO \div IO)^{\frac{1-n}{2}}$ will be greater than 1; and $(BO \div IO)^{\frac{1-n}{2}} - 1$, will be a positive quantity: but when n is greater than 1, or when the force varies in a ratio greater than that of the distances from quiescence, the rate will be retarded, because in that case $(BO \div IO)^{\frac{1-n}{2}}$ will be less than 1, and $(BO \div IO)^{\frac{1-n}{2}} - 1$ becomes negative. The converse of these propositions is likewise derived from the general Theorem.

Whenever, therefore, it is found, by observing the rate of a time-keeper, that a diminution of the arc of the balance's vibration causes an acceleration of the daily rate, it is necessary to conclude, that the elastic force of the spring in this case varies in a ratio less than that of the distances from the quiescent position. In like manner, when a diminution of the arc of vibration causes a retardation of the rate, it is certain that the spring's elastic force varies in a higher ratio than that of the distances from quiescence. It appears, indeed, from some experiments, that the weights which counterpoise a spiral spring's elastic force, when wound to different distances from the quiescent position, are in the ratio of those distances; but it is shewn from this proposition, and the annexed table, that the differences between the weights, by which the ratio of the distances, and a ratio a little less is indicated, although far too small to be discoverable by experiment, are yet sufficient to create a material alteration of the daily rate.

to

to be regulated to mean time when the ſemiarc of vibration is 135°, and let this ſemiarc be diminiſhed 8°, ſo as to become 127°; let the ratio of the ſpring's elaſtic force deviate from that of the diſtances from the quieſcent poſition by a ſmall difference of $\frac{1}{1000}$ part, ſo that the ſpring's force ſhall be in the $\frac{999}{1000}$ power of the diſtances, inſtead of in the entire ratio of the ſaid diſtances from the quieſcent poſition. The alteration of the daily rate of the watch will be obtained from the preceding theorem, by making the following ſubſtitutions. BO = 135°, IO = 127°, $n = \frac{999}{1000}$: the alteration of the daily rate $= 24^h \times ((135 \div 127)^{\frac{1}{2000}} - 1) = + 2''{\cdot}62$.

It here appears, that a very minute alteration in the law of the force's variation, amounting to no more than $\frac{1}{1000}$ part of the entire ratio of the diſtances, cauſes an acceleration in the daily rate of more than $2\frac{1}{2}''$, when the diminution of the ſemiarc of vibration is 8°. It may therefore be of ſome uſe to inquire, what are the differences of the weights to be obſerved in experiments from which the law of the ſpring's elaſtic force is derived; firſt, ſuppoſing that law to be the preciſe ratio of the diſtances from the quieſcent poſition; and ſecondly, ſuppoſing the law of the force to deviate from that ratio by a ſmall difference of $\frac{1}{1000}$, ſo as to become the $\frac{999}{1000}$ power of the diſtances from the quieſcent poſition; from the reſult a judgment may be formed how far experiments may be relied on for aſcertaining the preciſe law according to which the elaſtic force of a ſpring varies.

Values of x°. Angular distances from the quiescent position when the spring's elastic force is counterpoised by the weights in the second and third columns. (Fig. 80, Pl. 5.)	Values of $P = 9\,gr. \times x^\circ \div 90^\circ$ Weights P, expressed in grains, which counterpoise the spring's elastic force when wound to the several distances from the quiescent position in the first column, if the force varies in the precise ratio of the angular distances from the quiescent point.	Values of $P = 9\,gr. \times (x^\circ \div 90^\circ)^{\frac{999}{1000}}$ Weights P, expressed in grains, which counterpoise the spring's elastic force when wound to the several distances from quiescence in the first column, if the force varies in the $\frac{999}{1000}$ power of the distances from the quiescent point.*
	Grains.	Grains.
10°	1	1·002199
20°	2	2·003010
30°	3	3·003298
40°	4	4·003245
50°	5	5·002940
60°	6	6·002433
70°	7	7·001759
80°	8	8·000942
90°	9	9·000000

* It may here be observed, that the differences of the weights in the second and third columns of the table, first increase, and afterwards decrease; their difference is the greatest when the quantity $9 \times (x \div 90)^{\frac{999}{1000}} - 9x \div 90$ is a maximum; or when $x = 90^\circ \times (999 \div 1000)^{1000}$, that is, when $x = 33^\circ\ 5'\ 33''$.

The

The differences of weights expreſſed in the ſecond and third columns of this table, are evidently too ſmall to admit of being obſerved experimentally, and yet their effect on the daily rate of a time-keeper amounts to a quantity far from inſenſible. This effect on the rate might probably be augmented to twenty or thirty ſeconds daily, and yet the correſponding differences of weights ariſing from the deviation of the ſpring's force from the law of iſochroniſm might be too minute to become ſenſible by any ſtatical counterpoiſe of the ſpring's forces; and it would be ſtill leſs poſſible to meaſure the ſaid differences of weights with the exactneſs required for the determination of the law obſerved by the ſpring's forces. Experiments of this kind ſhould not therefore be abſolutely relied on for aſcertaining practically the iſochronal property of ſpiral ſprings, although this property muſt be allowed in theory, whenever the forces of elaſticity at the ſeveral angular diſtances from the quieſcent poſition are in the preciſe ratio of thoſe diſtances. The iſochronal law of variation, here mentioned, may be conveniently aſſumed in theoretical investigations, and proper corrections or equations, when neceſſary, may be applied to compenſate for the deviation from this law, which may ſubſiſt in any particular ſpiral ſpring, whenever it can be ſatisfactorily aſcertained, or reduced within the known limits, by ſuch mode of inference as the nature of the caſe may admit of. This aſſumption will appear the leſs exceptionable from conſidering, that the elaſtic forces of ſpiral ſprings which are not iſochronal deviate from the law of variation in queſtion, in ſome caſes by exceeding, and in others, by falling ſhort of it; and no other law is ſuggeſted either by theory or experiment, which more generally correſponds with the action of balance ſprings.

(To be continued.)

ARTICLE LI.

On the Reſolution of Indeterminate Problems.

By John Leſlie, A. M.

(Continued from page 320.)

Caſe 2. WHEN the third term is a ſquare. Suppoſe $C=c^2$, and the expreſſion is $ax^2+bx+c^2=y^2$. By tranſpoſition, $ax^2+bx=y^2-c^2$, and by reſolution, $(ax+b)x=(y+c)(y-c)$; whence by aſſumption, $x=(y+c)\div m$, and $ax+b=my-mc$. But from the ſecond equation, $x=(my-mc-b)\div a$, conſequently, $(my-mc-b)\div a=(y+c)\div m$; whence $y=(m^2c+mb+ac)\div(m^2-a)$, and $x=(y+c)\div m=(2mc+b)\div(m^2-a)$.

Suppoſe $3x^2+5x+16=y^2$, and $m=2$; then $x=(16+5)\div(4-3)=21$, and $y=(16+10+12)\div(4-3)=38$. But $3\cdot(21)^2\times 5\cdot 21+16=1444=(38)^2$.

Cor. 1. Let $b=0$; then the expreſſion becomes $ax^2+c^2=y^2$, and $x=2mc\div(m^2-a)$, and $y=(m^2c+ac)\div(m^2-a)$. Thus $2x^2+9=y^2$; if $m=2$, $x=4\cdot 3\div(4-2)=6$, and $y=(4\cdot 3+7\cdot 3)\div(4-2)=9$. But $2\cdot(6)^2+9=81=(9)^2$.

Cor. 2. If $b=0$, and $c=1$; then $ax^2+1=y^2$, and $x=2m\div(m^2-a)$, and $y=(m^2+a)\div(m^2-a)$. Put $a=m^2-d$, and we ſhall obtain $x=2m\div d$, and $y=(2m^2-d)\div d$. Hence it is evident, that x and y will be expreſſed in whole numbers, when $2m$ is diviſible by d. Call the quotient n; then $x=n$, and $y=mn-1$; whence $x\div y=(mn-1)\div n=m$
—1

$+1 \div n$, or $m-d \div 2m$, which are the two first terms of the *continued fraction* denoting $\surd(m^2-d)$, or $\surd a$. Thus, if $12x+1=y^2$; then $\surd 12 = \surd(16-4) = 4 - \cfrac{1}{2 - \cfrac{1}{2\ \&c.}}$ and $x=2$, and $y=4\cdot 2-1=7$; for $12\cdot 4+1=49=(7)^2$.

It is to be remarked, that, when $d=1$, the values of x and y may be discovered from any given number of terms of the continued fraction.

Thus, if $3x^2+1=y^2$; then $\surd 3=\surd(4-1)=2-\cfrac{1}{4-\cfrac{1}{4\ \&c.}}$

whence $x=4$, 15, 56, 209, &c. and $y=7$, 26, 97, 362, &c.

If $a=m^2+d$, then $x=-n$, and $y=-mn-1$; but the expression $ax^2+1=y^2$, will not be altered by changing the signs of x and y; whence $x=n$, and $y=mn+1$; consequently x and y will be determined from the continued fraction

$m+\cfrac{1}{n+\cfrac{1}{n+\&c.}}$ denoting $\surd(m^2+d)$. Thus, $20x^2+1=y^2$; then $\surd 20=\surd(16+4)=4+\cfrac{1}{2+\cfrac{1}{2\ \&c.}}$ and

$x=2$, and $y=4\cdot 2+1=9$; for $20\cdot 4+1=81=(9)^2$.

We may observe, that if $d=1$, the values of x and y, in the expression $(m^2+1)x^2\pm 1=y^2$, may be found by taking an even or odd number of terms, according as the sign $+$ or $-$ is to be adopted.

Cor. 3. Let $c=0$, then $ax^2+bx=y^2$; and, in this case, $x=b\div(m^2-a)$, and $y=mb\div(m^2-a)$. Thus, $7x^2+4x=y^2$; if $m=3$, then $x=4\div(9\ \ 7)$

$=2$,

$=2$, and $y=3\cdot4\div(9-7)=6$. For $7\cdot(2)^2+4\cdot2=36=(6)^2$.

Case 3. When BB—4AC is a square.

Let $x^2+(b\div a)x+c\div a=\text{D}\times\text{E}$; then the divisors of ax^2+bx+c will be $(a\div n)\text{D}$, and $n\times\text{E}$. But it appears, from the doctrine of equations, that the excesses of x above the roots of the quadratic, $x^2+(b\div a)x+c\div a=0$, are the divisors of the expression $x^2+(b\div a)x+c\div a$. Wherefore, $\text{D}=x+(b+\sqrt{b^2-4ac})\div2a$, and $\text{E}=x+(b-\sqrt{b^2-4ac})\div2a$. Hence, when $\sqrt{(b^2-4ac)}$ is a whole or fractional number, the expression ax^2+bx+c admits of resolution, and the divisors are $\frac{a}{n}\left(x+\frac{b+\sqrt{(b^2-4ac)}}{2a}\right)$ and $n\ (x+\left(\frac{b-\sqrt{(b^2-4ac)}}{2a}\right)$

And when these are found, the solution will be obtained from Prob. VI.

Suppose $14x^2+19x+6=y^2$, then $b^2-4ac=361-336=25$, and $\text{D}=\frac{14}{n}(x+\frac{19+5}{28})$, and $\text{E}=n(x+\frac{19-5}{28})$. If $n=2$, the divisors will be $\frac{14}{2}(x+\frac{6}{7})=7x+6$, and $2(x+\frac{1}{2})=2x+1$; whence from Prob. VI. $x=2$, and $y=10$. For $14\cdot4+19\cdot2+6=100=(10)^2$.

Case 4. When the general quadratic can be resolved into factors, if diminished by a given square.

Let $(ex+f)(gx+h)=y^2-d^2$, then $(ex+f)(gx+h)=(y+d)(y-d)$; whence $ex+f=my-md$, and $gx+h=(y+d)\div m$. By reducing the first equation, $x=(my-md-f)\div e$, and by reducing the second,

$x=$

$x=(y+d-mh)\div mg$; whence $(my-md-f)\div e=(y+d-mh)\div mg$, and consequently, $y=(m^2dg+mfg+de-meh)\div(m^2g-e)$. But $x=(my-md-f)\div e$, therefore also $x=(2md-m^2h+f)\div(m^2g-e)$.

Suppose $14x^2+31x+24=y^2$; then, taking $9=d^2$ from both sides, $14x^2+31x+15=y^2-d^2$; but $\sqrt{(b^2-4ac)}=\sqrt{(961-840)}=11$; whence, if $n=2$, the divisors Case III. will be $7x+5$ and $2x+3$; wherefore, making $m=2$, $x=(12-12+5)\div(8-7)=5$, and $y=(24+20+21-42)\div(8-7)=23$. For $14\cdot25+31\cdot5+24=529=(23)^2$.

(To be continued.)

ARTICLE LII.

Useful Propositions in Geometry.

By Mr. M. A. Harrison.

(Continued from page 285.)

PROP. V. THEO. *Fig.* 175. *Plate* 12.

THINGS remaining as in the last Proposition, I say that, the rect. LNF will be equal to the rect. LDK.

Dem. By parallels DC or LN : LE :: DK : KL, that is, LNF : LE·NF or LES :: LDK : DLK; but LES=DLK; therefore LNF=LDK.

Q. E. D.

Cor. If DV be joined, the rect. LNF will be = the square of DV^2.

PROP.

PROP. VI. THEO.

The lines being drawn as in the former propoſitions, I ſay that, the rectangle LFS will be equal to the ſquare of CP, that is, equal to the ſquare of half the ſum of the ſides.

For, $AP : LD :: LB : PC$,
that is, $AP^2 : LD^2 :: LB^2 : PC^2$:
but (Prop. IV.) $AP^2 = LES$, and $LD^2 = FSE$, and $LB^2 = FLE$;
therefore $LES : FSE :: FLE : PC^2$,
that is, $LES : LFS :: LES : PC^2$;
theref. the rectangle LFS is = to the ſquare of CP.

Q. E. D.

Cor. The rectangle contained by LF, EN is alſo = CP^2.
Cor. 2. The rectangle ECQ is equal to the rectangle LFS.
Cor. 3. The rectangle ACB is = to $PC^2 - PA^2$.
Cor. 4. The rect. LEN is = to the ſquare of EP.
Cor. 5. $EP^2 - PI^2$ is equal to $LS \cdot EF$.
Cor. 6. The rectangle IGE is = to the rectangle contained by AP, EF.

PROP. VII. THEO.

If EP be produced as in Prop. III. *Cor.* 2, I ſay that, the rectangle contained by 4EP, PI will be $= AD^2 - DB^2$.

Demon. It is well known that $AD^2 - DB^2$ is = to $AC^2 - CB^2$, that is = to $(AC + CB).(AC - CB)$, that is = to $2PC \cdot 2PA$, that is = to $4PC \cdot PA$, that is = to $4EP \cdot PI$.

Q. E. D.

Cor. 1.

Cor. 1. The rectangle EPI is = to the rectangle DLB.

Cor. 2. If IF be joined, GC (=BC) will be parallel and equal to it.

Cor. 3. Also GI will be parallel and equal to CF.

Cor. 4. Hence AI being joined, $AI^2 = GI^2 =$ EFN.

Cor. 5. If QP be produced to meet AI in n, then An=nI=nP.

Cor. 6. The angle AnP is equal to the angle ACB.

Cor. 7. The ∠ InP is = to the sum of the angles CAB, CBA.

PROP. VIII. THEO.

If W be the point of contact of the inscribed circle, then I say that, the rectangle AWB will be = to the rectangle ELN.

Demon. It is well known that LW is equal to LV, therefore, with the centre L and the radius LW, describe the semicircle WVY, then DV being joined, will touch it at V.

Now $AWB = ADB + DW^2 + 2DWL = ADB + KDL$,
but $ADB = NLS$, and $KDL = LN \cdot SE$;
theref. $AWB = NLS + LN \cdot SE = (LS + SE) \cdot LN = ELN$.

Q. E. D.

Cor. 1. The rectangle YDW is = to the rectangle LDK.

Cor. 2. The difference of the rectangles AWB, ADB, is equal to the rectangle LNF, that is, equal to the square of DV.

(To be continued.)

ARTICLE

ARTICLE LIII.

Three Propofitions from *Lawfon.*

(To be anfwered in Number VII.)

PROP. XXV.

LET AB be the diameter of a circle, and CD perpendicular to the fame, meeting the circumference in C and D, and let E be the centre, and from C and D let CF, DF, be inflected to any point F in the circumference, meeting the diameter AB in G and H; I fay the rectangle GEH is equal to the fquare of the radius AE.

PROP. XXVI.

In AB, the diameter of a circle, let two points C and D be taken, fuch that AC : CB :: AD : DB, and let D be without the circle, and DE perpendicular to DB; through the point C let any line be drawn meeting the circumference in F and G, and from the points F and G, let FH and GH be inflected to any point H in the circumference, meeting DE in K and L; I fay the rectangle KDL is equal to the rectangle ADB.

PROP. XXVII.

In AB, the diameter of a circle, let be taken the point C, and DE be perpendicular to AB, meeting the circumference in D and N; in CD let be taken

taken two points E and F on the ſame ſide of C with D, ſuch, that the rectangle ECF may be equal to the ſquare of CD; and from the points E and F let EG, FG, be inflected to any point G in the circumference, meeting the ſame in H and K; and let HK, when drawn, meet the diameter AB in L; then I ſay that AL : LB :: AC : CB.

ARTICLE LIV.

Two Propoſitions from *Stewart*'s Theorems.

(To be anſwered in Number VII.)

PROP. XXIII. THEO. XX.

LET there be any regular figure circumſcribed about a circle of a greater number of ſides than three, and from any point within the figure let there be drawn perpendiculars to the ſides of the figure, and likewiſe let there be drawn a right line to the centre of the circle; twice the ſum of the cubes of the perpendiculars drawn to the ſides of the figure, will be equal to twice the multiple of the cube of the ſemidiameter of the circle by the number of the ſides of the figure, together with thrice the multiple by the ſame number of the ſolid, whoſe baſe is the ſquare of the line drawn to the centre, and altitude, the ſemidiameter of the circle.

PROP.

PROP. XXIV. THEO. XXI.

Let there be any figure given by position of a greater number of sides than four; four right lines may be found that will be given by position, such, that if from any point within the figure there be drawn perpendiculars to the sides of the figure, and likewise there be drawn perpendiculars to the four lines found, four times the sum of the cubes of the perpendiculars drawn to the sides of the figure, will be equal to the multiple of the sum of the cubes of the perpendiculars drawn to the four lines found by the number of the sides of the figure.

ARTICLE LV.

Demonstrations to Lawson's *Propositions proposed in* ARTICLE XXXI.

PROP. XVIII.

Demonstrated by Mr. Colin Campbell, Fig. 178, 179, Pl. 13.

JOIN AB. Since AB = AC, the arc AB = arc AC; conse. $\angle$BDA, in fig. 178, = $\angle$CBA, and in fig. 179, = sup. of $\angle$ACB = sup. $\angle$ABC = $\angle$ABE; wherefore the $\triangle$'s BDA, BAE, are equiangular; therefore DA : AB :: AB : AE,
and therefore DAE = AB^2.

Q. E. D.

The

The same by Mr. John Lowry.

Join DC. Since AB=AC, the ∠ACB=∠ABC=∠ADC; therefore the Δ's AEC, ACD, are equiangular;
therefore AC : AE :: AD : AC,
and therefore $DAE = AC^2 = AB^2$.

Q. E. D.

The same by Mr. Richard Nicholson.

Draw the diameter AL, and let it meet BC in I. Join LD, LB. Because AC=AB,
the ∠ABC = ∠ACI = ∠ALB,
and the ∠BAL is common to the Δ's ABI, ABL;
therefore the ∠AIB=∠ABL=ADL;
wherefore the Δ's AIE, ADL, are equiangular;
therefore AL : AD :: AE : AI,
therefore IAL=DAE, and by Eu. VI. 8, $IAL = AB^2$;
wherefore $DAE = AB^2$.

Q. E. D.

The same by Mr. J. H. Swale.

ANALYSIS.

Join BD. By hypothesis $DAE = AB^2$,
that is, DA : AB :: AB : AE;
therefore the Δ's ABE, ABD, are equiangular:
wherefore the ∠ABC = ∠EDB = ∠ACB,
therefore AB is equal to AC.

Q. Q. V.

PROP. XIX.

Demonstrated by Messrs. Campbell *and* Lowry.
Fig. 178, 179, Pl. 13.

By Eu. III. 32. The ∠FAC=∠ABC=∠ACB,
therefore AF, BC, are parallel,

 and

and therefore the $\triangle$'s ACB, ACF, are equiangular;
wherefore $AC : BC :: AF : AC$,
theref. $AC^2 = BC \cdot AF$, and Prop. XVIII. $AC^2 = DAE$;
therefore $BC \cdot AF = DAE$;
wherefore $BC : AE :: AD : AF$:
but by parallels, $AD : AF :: DE : EG$;
therefore $BC : AE :: DE : EG$;
therefore $BC \cdot EG = DEA = BEC$,
wherefore $BC : BE :: EC : EG$,
therefore $BC : EC :: EC : CG$,
and therefore $BCG = EC^2$.

Q. E. D.

The same by Mr. Nicholson.

By my dem. to Prop. XVIII. the $\angle AIB = \angle ADL$; but ADL is a right $\angle$, therefore AIB is a right $\angle$; wheref. BC is perp. to AL, and AF is perp. to AL; therefore AF is parallel to BC and equal to FC, wherefore the $\triangle$'s AFC, ABC, are equiangular.

Mr. N. now proceeds exactly as Messrs. Lowry and Campbell have done above.

The same by Mr. J. H. Swale.

ANALYSIS.

By hypothesis $BCG = EC^2$,
that is, $BC : EC :: EC : CG$,
and by division $BC : BE :: EC : EG$;
therefore $BC \cdot EG = BEC = AED$,
wherefore $ED : EG :: BC : AE$.
Again, the $\angle FAC = \angle ABC = \angle ACB$;
and therefore AF is parallel to BC;
wherefore $ED : EG :: AD : AF$;
therefore $BC : AE :: AD : AF$,
wherefore $BC \cdot AF = DAE$:
but Prop. XVIII. $DAE = AB^2 = AC^2$;
therefore $BC \cdot AF = AC^2$,

wherefore

re BC : AC :: AC : AF;
e the Δ's ACB, ACF, are equiangular,
'efore ∠ACF = ∠ABC = ∠FAC;
re CF is equal to AF.

Q. Q. V.

PROP. XX.

rated by Messrs. Campbell, Lowry, Nichol-
, *and* Swale. *Fig.* 174, 176, *Plate* 12.

ANALYSIS, by Mr. Swale.

'pothesis FG·P=BFC=AFE;
e P : AF :: FE : FG :: AE : AD;
re AD·P=EAF:
). XVIII. EAF=AB^2;
e AD·P=AB^2.

Q. Q. V.

HESIS, by Messrs. Campbell, Lowry, *and* Nicholson.

'pothesis AD·P=AB^2,
'rop. XVIII. EAF=AB^2.
e AD·P=EAF;
re P : AF :: EA : AD :: FE : FG;
e P·FG=AFE=BFC.

Q. E. D.

PROP. XXI.

rated by Mr. Campell. *Fig.* 177, *Plate*
Plate 12, *and Fig.* 181, *Plate* 13.

: AB in O, and join EO, LO, EC, GO.
. I. CDO=ADB=DE^2;
DCO(=CDO+CD^2)=DE^2+CD^2=CE^2:

 but

but, Prop. XV. CE is a tangent to the circle at E; theref. $GCH = CE^2$, and theref. $DCO = GCH$. Hence, becaufe the ∠'s LGO, LDO, are right ones, the points L. G, O, D, H, will be in a circle; wherefore $EKF = GKH = LKD$,
that is $(DE - DK)\cdot(DE + DK)$ or $DE^2 - DK^2 = LKD$;
hence $(LK + KD)\cdot KD$, that is, $LDK = DE^2$.

Q. E. D.

The fame by Mr. Lowry.

Draw HD to meet the circle in Q, and join GQ.
By hypothefis $AC : CB :: AD : DB$;
theref. Conv. Prop. VII. GQ is parallel to EF;
wheref. Eu. III. 32. $\angle LGH = \angle GQH = \angle QDF = \angle HDK$;
therefore the △'s HKD, LKG, are equiangular,
and therefore $LKD = HKG = EKF$,
or $LKD + DK^2 = EKF + DK^2$,
that is, $LDK = EKF + DK^2$;
but $EKF = (ED + DK)\cdot(ED - DK) = ED^2 - DK^2$;
therefore $LDK = DE^2$.

Q. E. D.

The fame by Mr. Nicholfon.

To the centre O draw LO, interfecting GH in P, and join EO, OH, LH.

Then becaufe of the right ∠'s OGL, ODL, a circle will pafs through the points G, O, D, L, and by Prop. XV. CE is a tangent to the circle at E; therefore $GCH = ACB = CE^2$:
but, by the rt. ∠'d △COE, $DCO = CE^2$;
therefore $DCO = GCH$,
and therefore the point H is in the circle that paffes through the points G, O, D, L; hence the right ∠'d △'s OGL, OHL, having $OG = OH$, and OL common, will likewife have $LG = LH$, and therefore $HP = PG$;
wherefore, LO is perpendicular to HG;

wheref.

wheref. the $\triangle$'s LQK, LDO, CDK, are equiangular,
theref. $LDK = CDO$: but Eu. VI. 8. $CDO = DE^2$;
therefore $LDK = DE^2$.

Q. E. D.

The same by Mr. Swale.

ANALYSIS.

To the centre O draw LO, meeting GH in P.
By hyp. $LDK = DE^2$; add DK^2 to each,
then $LKD = DE^2 + DK^2 = EKF = GKH$;
theref. the points O, L, G, D, H, are in a circle whose diameter is OL ;
theref. the right $\angle$'d $\triangle$'s LOG, LOH, having $LG = OH$, and LO common, will also have $LG = LH$, and the $\angle GLO = \angle HLO$;
theref. GH is bisected in P, and OL is per. to GH ;
wheref. the $\triangle$'s LPK, LDO, are equiangular ;
hence $CDO = LDK = DE^2$;
theref. Eu. VI. 8. the $\triangle$ CEO is right $\angle$'d at E ;
wheref. CE touches the circle in E ;
theref. Conv. Prop. XV. AC : CB :: AD : DB.

Q. Q. V.

ARTICLE LVI.

Answers to the Mathematical Questions proposed in ARTICLE XXXIII. No. III.

I. QUESTION 49, *answered by Mr.* John Lowry.

IN figure 182, plate 13, take AB = the given distance of the tops of the mountains, and bisect it in D, and draw DE perpendicular thereto and = to the sum of the earth's radius and the nearest distance between the surface of the earth and the line connecting their tops; divide BD in Q, so that the rect. QBD may be = the square of half the given difference of the heights of the mountains; join QE, and draw BG parallel thereto, meeting DE produced in G; through the three points A, G, B, describe a circle, and let it meet EC, drawn parallel to AB, in C, and join AC, CB; with the centre C and distance = the earth's radius, describe the arch KHI; and AK, BI, will be the heights of the mountains required.

For, Cb being drawn perpendicular to AB, it is evident that Hb is equal to the nearest distance between the surface of the earth and the line connecting the tops of the mountains, and AB is = to the distance between their tops. Let DG be produced to meet the circle in F, and upon CB let fall the perpendicular FP; join BF, CF, CG.

Then by sim. △'s $BP:CG::BF:FG::BD:BG$,
alternating, $BP:BD::CG:BG$;
ther. $BP^2:BD^2::CG^2:BG^2::EGF:DGF::EG:DG$:
but $ED:DQ::GD:BD$,
and by alter. and divis. $GE:GD::BQ:BD::QBD:BD^2$;
theref. by equ. $BP^2:BD^2::QBD:BD^2$;

wherefore

wherefore $BP^2 = QBD$: but $2BP = BC - AC$; theref. $IB - IK = BC - AC =$ the given difference.

Q. E. D.

The method of calculation is evident from the conſtruction.

And thus the anſwer is given by Meſſrs. Nicholſon, Swale, *and* Thornoby.

Otherwiſe by Mr. Ralph Simpſon.

Let KHI repreſent the ſurface of the earth, C its centre, and A, B the two mountains; join AC, CB, and draw Cb perpendicular to AB, the line joining the tops of the mountains, then Hb is evidently the neareſt diſtance between the ſurface of the earth and the line connecting the tops of the mountains, which is given by the queſtion, and $CH = CK = CI =$ the earth's radius is given; therefore in the $\triangle ACB$ there is given the baſe AB, the $\perp$ Cb, and CB—CA. Hence, (Simp. Trig. Prop. XIV.) $2\, AB \cdot Cb : (AB + \overline{CB - CA}) \cdot (AB - \overline{CB - CA}) ::$ radius : tang. of $\frac{1}{2} \angle ACB$; therefore the $\angle ACB$ is given.

And (ibid. Prop. VIII.) $AB : CB - CA ::$ coſine $\frac{1}{2} \angle ACB$: the ſine of half the difference of the $\angle$'s CAB, CBA; therefore the ſum and difference of the $\angle$'s CAB, CBA, are given, and conſequently the $\angle$'s themſelves are given, from whence the ſides CA, CB (and conſequently the heights of the mountains IB, AK), may eaſily be determined.

The ſame algebraically by Mr. Johnſton.

Who puts $CB - CA = d$, $AD = \frac{1}{2}AB = a$, $Cb = b$, and $Db = x$, then will $Ab = a - x$, $Bb = a + x$;

wheref.

wheref. $AC=\sqrt{(b^2+\overline{a+x}|^2)}$, and $BC=\sqrt{(b^2+\overline{a+x}|^2)}$; theref. $\sqrt{(b^2+\overline{a+x}|^2)}-d=\sqrt{(b^2+\overline{a-x}|^2)}$, and by reduction, $x=\sqrt{(4d^2\cdot(a^2+b^2)-d^4)\div(16a^2-4d^2)}$. Hence the heights of the mountains may be found.

Messrs. Lee *and* Wood *answered it nearly in the same manner.*

II. QUESTION 50, *answered by Mr.* Lowry.

In figure 183, plate 13, let the primitive circle ZRNH represent the meridian of the required place, HR the horizon, ZN the prime vertical, PS the axis, EQ the equator, mno the parallel of declination for the given day, and A and B the sun's places when the observations were made. Conceive a great circle to be described through the points A and B. Then in the isosceles △ ABP, there is given AP=BP=74° 55′ 19″ the co-declination, ∠ ABP=30°=the difference of the times, to find AB =28° 56′ 38″. And in the △ AZB, (by comparing art. 221 and 241 of Wales' Robertson's Navigation,) it will be as the versed-sine of the ∠ AZB (=23° 29′ 42″ the difference of the azimuths) is to radius, so is the difference of the versed-sines of AB and AZ—BZ (=10° the difference of the altitudes,) to half the difference of the versed-sines of AZ+BZ, and AZ—BZ=·6606692 (radius unity); therefore AZ+BZ=109° 39′ 56″, and therefore AZ=59° 49′ 58″, and BZ=49° 49′ 58″.

Again, in the △ PAZ there is given AP, AZ, and ∠ AZP=108° 43′ to find the hour angle APZ= 57° 59′ 56″ and the co-latitude PZ=34° 9′ 35″; hence the times from noon are $1^h\ 51^m\ 59^s$ and $3^h\ 51^m\ 59^s$, and the latitude 55° 50′ 25″ North.

The

The same by the Rev. Mr. L. Evans.

Let Z be the zenith, P the pole, and A, B the sun's places at the required times of observation. Draw the great circles AP, BP, AZ, BZ, and BA, then there is given AP=BP, ∠ABP= the difference of times, AZ—BZ= the difference of altitudes, and ∠ AZB = the difference of azimuths. Find AB, and then by the last proposition of Simpson's Trigonometry, Cor. 3, (putting *V* for the versed-sine of the ∠AZB, and W for the versed-sine of its supplement), we have cosine of AZ+BZ=(2R× cosine AB—W× cosine (AZ—BZ))÷V; hence AZ+BZ and AZ—BZ being given, AZ and BZ will be given; whence every thing else is easily had.

Otherwise by Mr. Bulmer.

Who finds AB=28° 56′ 36″, and then puts d= versed-sine of AZ—BZ, f= versed-sine of AZ+BZ, v= versed-sine of ∠ AZB, and n=versed-sine of AB. By *Theo. X. page* 101, *of Thacker's Problems*, we have the equation, $d+\frac{1}{2}(fv-dv)=n$, and $f=(2n+dv-2d)\div v$: but 2—v is the versed-sine of the supplement of the ∠AZB, which call w, and then $(2n-dw)\div v=f$. Hence the sum of AZ, BZ =109° 39′ 44″, the half of which, viz. 54° 49′ 52″, added to half their difference = 5°, gives AZ= 59° 49′ 52″, and the difference between the said half sum and half difference gives BZ=49° 49′ 52″. Whence the latitude is easily found = 55° 51′ 24″, and the times when the observations were made are found to be 8 min. past 8, and 8 min. past 10 o'clock.

It was answered also by Messrs. Harris, Simpson, *and* Swale.

III. *QUESTION* 51, *answered by Mr.* Richard Nicholson.

Produce CD (fig. 184, pl. 13.) if necessary, to meet a parallel to BC, drawn from A, in L; draw KM parallel to BC.

Take AN : NL as Be : eC, that is, as AE : ED. Then NE being joined, will be parallel to LC; join Ne, and let it meet KM in P, then by parallels we have LD : NE :: AL : AN :: KM : KP :: MH to a parallel to LC, drawn from the point P, to terminate in KH, that is, by alternation, LD : MH :: NE : the aforesaid parallel.

Again, by parallels, and the proposition, it will be as CL : CM :: BA : BK :: CD : CH; therefore, by alternation and division, LD : MH :: CD : CH; and by parallels and equality, LD : MH :: BA : BK :: eN : eP :: NE to a parallel to LC, drawn from P, to terminate in Ee. Hence, by equality, as NE : the first-mentioned parallel :: NE : to the last-mentioned parallel; therefore those parallels are equal, and they are drawn from the same point; therefore they can meet Ee and KH in no other point but that of their intersection; and therefore, are coincident; hence, Be : eC :: KP : PM :: KO : OH.

In the same way it may be shewn that KH will divide any number of lines in the same ratio, if drawn according to the proposition.

Ingenious solutions were sent by Messrs. Lowry, Swale, *and* Thornoby.

IV. *QUESTION* 52, *answered by Mr.* Harris.

In this question, for " one third part," read " 30 square inches," and the solution will be as follows: —By hydrostatics the weight of the globe will be to

to the weight of the ſegment immerſed, as the ſpecific gravity of water to the ſpecific gravity of oak, that is, as 1000 to 925; therefore the weight of the globe will be to the weight of the ſegment above the water, as 1000 : 75.

Put x for the diameter of the globe, and y for the verſed-ſine, or height of the ſegment above the water.

By menſuration, $x^3 : (3x-2y) \cdot y^2 :: 1000 : 75$, and by the queſtion, $3{\cdot}14159 \times yx = 30$

From this equation $x = 30 \div 3{\cdot}14159\, y = m \div y$; hence, by ſub. $m^3 \div y^3 : (3m \div y - 2y) \cdot y^2 :: 1000 : 75$, whence, by reduction, $y = 1{\cdot}2652$ and $x = 7{\cdot}5486$ inches, the diameter of the globe required.

Otherwiſe by Maſter John Grier, *Pupil to Mr.* Harris.

Put $x =$ the verſed-ſine or height of the ſegment unimmerſed, and $nx =$ the diameter of the globe. Then by menſuration and the ſpecific gravity of bodies $n^3x^3 : (3nx-2x) \cdot x^2 :: 1000 : 75$, or $n^3 : 3n-2 :: 1000 : 75$, and $75n^3 = 3000n - 2000$; hence, $n = 5{\cdot}97$, and therefore, let the magnitude of the globe be what it will, provided it be ſolid oak, and the fluid common water, the height of the ſegment unimmerſed will be to the diameter of the globe in the conſtant ratio of $1 : 5{\cdot}97$.

Again, by menſuration and the queſtion

$3{\cdot}1416 \times nx^2 = 30$, & $x = \sqrt{30 \div (3{\cdot}1416 \times n)} = 1{\cdot}265$; hence, $nx = 7{\cdot}548$ the diameter of the globe.

Anſwered alſo by Meſſrs. Bulmer, Lowry, Simpſon, *and* Swale.

V. QUES.

V. QUESTION 53, *answered by Mr.* John Lee.

Let ABC (fig. 185, pl. 13.) represent the conical sugar loaf, D the top of the candle, and APFH the shadow made by the cone upon the ceiling; draw the lines as in the figure, and let O be the centre of the circle HGPF made by the shadow of the base of the cone. Then there is given in feet, the altitude of the cone $=EL=1\frac{1}{2}$, BC the diameter of the cone's base $=1$, $CD=10$, and $DE=5\frac{1}{2}$, and therefore $DL=4$.

Now, by sim. Δ's, $DL : DE :: DC : DG$,
and $DC : DG :: BC : FG$;
therefore $DL : DE :: BC : FG = 11 \div 8$,
and by Eu. I. 47, $CL = \sqrt{CD^2 - DL^2} = 2\sqrt{21}$.
But $DL : CL :: DE : EG = 11\sqrt{21} \div 4$, and $AE = 2\sqrt{21} + \frac{1}{2}$;
therefore $AG = EG - AE = 3\sqrt{21} \div 4 + \frac{1}{2}$,
and $AO = AG + GO = 3\sqrt{21} \div 4 + 3 \div 16$;
hence $AP = \sqrt{AO^2 - PO^2} = 3.5593$.
Again, by sim. Δ's, $AO:OP::OP:OX = \cdot 13038$,
and $AO:OP::AP:PX = \cdot 675$;
therefore $FX = \cdot 81788$, and $AX = 3\cdot 49405$;
hence the area of the segment $HPF = \cdot 92354$
and the area of the triangle $APH = 2\cdot 35848$

theref. the area of the whole shadow is $= 3\cdot 282$ feet.

And thus it was answered by Messrs. Lowry, Simpson, Swale, *and* Thornoby.

VI. QUESTION 54, *answered by Mr.* Lowry.

In fig. 186, pl. 13, let ABMR represent a section of the sphere passing through the centre, and perpendicular to the sides of the hole, and EFIK, another

another fection paffing through the angular points E, F, K, I, of the hole on the furface of the fphere; draw the lines as in the figure, and join ZF, FY, with great circles; alfo conceive a great circle to pafs through the points F, I.

Then OF is evidently the fine of half the arch FI; but OE is $= \sqrt{18} =$ the fine of 45° to radius 6; theref. the arch FI$=$90°, and arch FS$=$IT$=$45°. Now Z is the pole of the circle DPC, and Y is the pole of the circle BR; theref. the arch ZY is$=$90°, and ZS, SY, each $=$ 45°, and by trig. ZF$=$FY$=$60°, and $\angle$ FZS$=$ZFS$=$54° 44′, theref. the fum of the $\angle$'s in the $\triangle$ FZS$=$199° 28′; hence, as 180° : 19° 28′ (199° 28′$-$180) :: a great circle of the fphere ($=$113·0976) : 12·2312 the area of the $\triangle$ ZFY.

Moreover, becaufe the arch ZF or ZC$=$60°, its verfed-fine is $=$ to half the rad. of the fphere $=$3; therefore, as the diameter 12 : 3 :: the furface of the fphere ($=$144$\times$3·14159) : the furface of the fegment DABC$=$36$\times$3·14159. Then as 360° : $\angle$ FZC($=$54° 44′) :: 36$\times$3·14159 : 17·195 $=$ the furface FZC, $=$ alfo to the furface YFB; therefore, from its double, take the $\triangle$ ZFY, and there remains 9·9276 $=$ the furface of the part CFB, the double of which drawn into one-third of the radius of the fphere is $=$ to the folidity of the fpheric fector BOC$=$39·7088.

Again, PC$=\sqrt{OC^2-OP^2}=\sqrt{27}=3\sqrt{3}$; therefore DC $=$ 10·3923048 and FC$=$2·1961424; hence the area of the fegment whofe verfed fine is CF and diameter of the circle DC, is$=$ 13·072273; therefore the folidity of the pyramid whofe bafe is this fegment, and altitude OP $=$ 3, is$=$ 13·072273, and the double of this taken from 39·7088, the folidity of the fector, leaves 13·5644 for the folidity of the fecond fegment CFBFC. But the content of

the

the ſegment DABC is$=141\cdot372$,and ſince the ſides of the ſquare hole are equidiſtant from the centre of the ſphere, the ſegments DABC, BCQR, QRMG, MGDA, are equal, as are likewiſe the ſecond ſegments BCF, QKR, MIG, DEA; therefore $4\times$ ſegment DABC $-4\times$ ſecond ſegment CFBFC$=$ $511\cdot23$ the ſolidity of the part remaining, which taken from $904\cdot78$, the ſolidity of the globe, leaves $393\cdot55$ inches for the ſolidity of the piece cut out.

N. B. For an illuſtration of this ſolution, ſee Dr. Hutton's excellent Treatiſe on Menſuration.

Mr. Thornoby *alſo pointed out a method of ſolution.*

VII. QUESTION 55, *anſwered by Mr.* Bulmer.

Let AP (fig. 187, pl. 13) repreſent the tower, PBC the horizontal plane, ADB the path of the ball before reflection, and BLC its path after reflection. Draw ADM parallel to PB, meeting BM drawn parallel to AE (the direction of the ball at the beginning of the motion) in M. Erect the perpendicular BH, biſect AD in O, and draw ROV, QDE, parallel to BH, and at the point B draw the tangents BS, BK, which, by the nature of reflection, will make the $\angle$'s SBQ, KBC, $=$ to each other.

Now by projectiles, ſine of twice $\angle$EAD $\times$ the ſquare of the velocity $\div$ed by $32\frac{1}{6}=\text{AD}=45117\cdot5$ feet, and by trigonometry as tang. $\angle$ HMB$=$ DAE$=20^\circ\ 5'$: radius :: BH$=80$: HM$=218\cdot8$. Again, by prop. 2d. of *Simpſon*'s Select Exerciſes, $\text{OH}=\sqrt{\text{AO}\cdot(\text{AO}+2\text{MH})}=22776\cdot5$, therefore PB $=45335\cdot25$.

Moreover VO$=$tang. $\angle$DAE$\times\frac{1}{4}$AD$=4123\cdot9$, therefore VR $=$ VS$=$ $4203\cdot9$ and SR $=8407\cdot8$. Hence in the right-angled $\triangle$ SRB there is given two

two fides SR, BR, to find the $\angle RBS = 10^\circ\ 27'\ 27''$ $= \angle KBC$. Again, by projectiles, the velocity of the ball at B is found $=1501{\cdot}7$ feet in one fecond, and, fine of twice $\angle KBC \times$ fquare of the velocity at B $\div$ed by $32\frac{1}{6} = BC = 25026{\cdot}9$ feet.

Whence the time the ball was in motion, its velocity at C, and PC, the diftance from the bottom of the tower may be eafily determined.

Meffrs. Lowry, Peacock, Simpfon, *and* Thornaby, *favoured us with ingenious folutions to this queftion.*

VIII. QUESTION 56, *anfwered by Mr.* W. Peacock.

Let v reprefent the velocity acquired by the ball in defcending from the balloon to the furface of the elm, n the fpace the ball would penetrate into the elm, impinging upon it, with that velocity; then the velocity being as the fquare roots of the fpaces, we have

$\sqrt{n} : \sqrt{n} - \sqrt{n-1} :: v : (\sqrt{n} - \sqrt{n-1}) \cdot v \div \sqrt{n} =$ the velocity loft by the ball in paffing through the 1ft inch of the elm;

$\sqrt{n} : \sqrt{n-1} - \sqrt{n-2} :: v : (\sqrt{n-1} - \sqrt{n-2}) \cdot v \div \sqrt{n} =$ do. in the 2d in.

$\sqrt{n} : \sqrt{n-2} - \sqrt{n-3} :: v : (\sqrt{n-2} - \sqrt{n-3}) \cdot v \div \sqrt{n} =$ do. in the 3d in.

$\sqrt{n} : \sqrt{n-3} - \sqrt{n-4} :: v : (\sqrt{n-3} - \sqrt{n-4}) \cdot v \div \sqrt{n} =$ do. in the 4th in.

hence $(\sqrt{n} - \sqrt{n-1}) \cdot v \div \sqrt{n} + (\sqrt{n-1} - \sqrt{n-2}) \cdot v \div \sqrt{n} + (\sqrt{n-2} - \sqrt{n-3}) \cdot v \div \sqrt{n} + (\sqrt{n-3} - \sqrt{n-4}) \cdot v \div \sqrt{n} = (\sqrt{n} - \sqrt{n-4}) v \div \sqrt{n} =$ the velocity loft when the ball has paffed through the four inches of elm. Therefore $v -$ $(\sqrt{n} \sqrt{n-4}) \cdot v \div \sqrt{n} = \sqrt{n-4} \cdot v \div \sqrt{n} =$ the velocity of the ball at that time. But by the queftion, this

velocity muſt be ſuch as will carry it through 60 feet in the next ſecond, that is, $\sqrt{n-4}\cdot v \div \sqrt{n} = 43\tfrac{1}{12}$; hence $v^2 = (43\tfrac{1}{12})^2 \cdot n \div (n-4)$, but, by Prob. 9, of Dr. *Hutton*'s Select Exerciſes $(1 \times 11\tfrac{1}{3} \times 13) \cdot v^2 \div (2 \times 7\tfrac{1}{3} \times \overline{1500}|^2) = n$, and $v^2 = n \cdot (2 \times 7\tfrac{1}{3} \times \overline{1500}|^2) \div (1 \times 11\tfrac{1}{3} \times 13)$; therefore, by equating theſe two values of v^2, we have $(43\tfrac{1}{12})^2 \cdot n \div (n-4) = n \cdot (2 \times 7\tfrac{1}{3} \times \overline{1500}|^2) \div (1 \times 11\tfrac{1}{3} \times 13)$, from which $n = 4 + (\overline{43\tfrac{1}{12}}|^2 \times 1 \times 11\tfrac{1}{3} \times 13) \div (2 \times 7\tfrac{1}{3} \times \overline{1500}|^2) =$ 4·006121, therefore $v^2 \div (4 \times 16\tfrac{1}{12}) = 13956 \cdot 46 =$ Lunardi's height above the block, and therefore his height above the earth $= 14016 \cdot 46$ feet

Solutions equally ingenious were received from Meſſrs. Bulmer, Lee, Lowry, *and* Swale.

IX. QUESTION 57, *anſwered by Mr.* Lowry.

Fig. 188, Pl. 13. Let ZAH repreſent a quadrant of the meridian, AH the altitude of the culmen cœli, DN the altitude of the nonageſimal degree, NZ its complement, NA perpendicular to DZ and equal to the diſtance of the nonageſimal degree from the meridian $= 16^\circ\ 44'$.

Since AH+DN is given $= 78^\circ\ 25'$, AZ+NZ, the ſum of their complements, will be given $= 101^\circ\ 35'$, and the $\angle$ ZNA is $= 90^\circ$; therefore if NZ be produced to Q, ſo that ZQ may be = ZA, and AQ be drawn, there will be given AN, NQ, and the $\angle$ZNA to find AQ$=101^\circ\ 5'\ 31''$, and the $\angle$NQA$=17^\circ\ 3'\ 40''$. Draw ZP perpendicular to AQ, then in the rt. $\angle$'d $\triangle$ APZ there is given AP $=\tfrac{1}{2}$ AQ$=50^\circ\ 32'\ 45''$, and the $\angle$PAZ = NQA, to find AZ$=51^\circ\ 48'\ 19''$; hence NZ $= 49^\circ\ 46'\ 41''$, and

and ND = $40° \, 13' \, 19''$ the altitude of the nonagesimal degree.

Ingenious solutions to this question were also received from Messrs. Dawes, Swale, *and* Thornoby.

X. QUESTION 58, *answered by Mr.* Lowry.

Let Z (fig. 189, pl. 13.) be the zenith, P the pole, and O the sun's place. Then there is given the ∠ $OZP = 93° \, 8' \, 30''$ (= the supplement of the given azimuth), $OP = 67°$, the comp. of the declination, and $OZ + ZP = 180° - 92° \, 4' = 87° \, 56'$. Hence, by the last Prop. and last Cor. of *Simpson*'s *Trigonometry*, putting V for the versed sine of the ∠ Z, and W for the versed sine of its supplement, we have cosine of $(OZ - ZP) = (2 \times \text{Rad.} \times \text{cos. } OP - V \times \text{cos. } (OZ + ZP)) \div W$ = cos. of $37° \, 56'$; hence $OZ = 62° \, 56'$, and $ZP = 25°$, and ∠ $OZP = 75°$; therefore the required latitude is 65°, and the hour five P. M.

Messrs. Dawes, Swale, *and* Thornoby *likewise sent ingenious solutions to this question.*

XI. QUESTION 59, *answered by Mr.* Lowry.

The figure remaining as in the last question, we have given in the △ OZP, the ∠ $OZP = 96° \, 19'$ (the sup. of the given azimuth), the hour ∠ $OPZ = 68° \, 45'$, and $OZ + ZP + PO$ = 3 quadrants $- 111° \, 45' = 158° \, 15'$, to find ZP. Continue ZP, ZO till ZB, ZA, be each = half the sum of OZ, ZP, PO; draw the perpendicular arches BC, AC, to meet in C, and draw CZ, CP, which, as is well known, will bisect the ∠'s OZP, OPB. Hence, in the rt. ∠'d △ CZB, we have $ZB = 79° \, 7' \, 30''$, and ∠ $CZB = 48° \, 9' \, 30''$ to find $CB = 47° \, 38' \, 30''$, and in the rt. ∠'d △ CBP there will be given, $CB = 47° \, 38' \, 30''$, and ∠ $CBP = \frac{1}{2}$ the sup. of ∠ $OPZ = 55° \, 37' \, 30''$,

 to

to find BP = 48° 36′ 39″, which taken from ZB, leaves ZP = 30° 30′ 51″ the complement of the latitude.

And thus the anſwer is given by Meſſrs. Dawes, Swaie, *and* Thornoby.

XII. QUESTION 60, *anſwered by Mr.* Lowry.

Here, in fig. 189, pl. 13, we have given in the △ OZP, the co-lat. ZP, the hour ∠ OPZ, and OZ+OP, to find OZ the co-altitude, which may be determined, as in queſtion 58, when the ſum of the declination and altitude is conſiſtent with the reſt of the data, but in the preſent inſtance it is given conſiderably too large.

And thus it was anſwered by Meſſrs. Dawes, Swale, *and* Thornoby.

XIII. QUESTION 61, *anſwered by Mr.* Lowry.

ANALYSIS.

Suppoſe the problem ſolved, and that LC, LA, (fig. 190, pl. 13.) drawn at any convenient angle to each other, are the lines required, and having the given ratio; take LO = LG = to the given line. Join AC and draw GH parallel thereto.

By parallells LA : LC :: LG : LH ;
but LG, and the ratio of LA to LC are given; therefore LH and conſequently OH will be given. Now by the Prob. OC·GA is given, and the ratio of GA to HC is given, being the ſame as the ratio of LA to LC ; therefore the rect. OCH is given. Hence the Prob. is reduced to this, *viz.* to produce OH to C, ſo that the rect. OCH may be equal to

a

a given ſpace, which is elegantly done in *Profeſſor Playfair's* edit. of the Elements. III. 29.

The ſame by Mr. Rd. Nicholſon, Leeds.

Conſ. fig. 190, pl. 13, take GH = to twice the given line, and biſect it in F; draw FB ⊥ to GH and = to the ſide of the given ſquare; divide GH at K in the given ratio; join BK and produce it to meet a ſemicircle deſcribed upon GH in L, join LG, LH, and produce them till they meet a parallel to GH, drawn through B, in A, C; AB, BC, are the lines required.

Demon. Draw FD, FE parallel to LA, LC. By ‖'s and *Conſ.* GF=AD=FFH=FC=the given line,
and GK : KH :: AB : BC, the given ratio,
and the ∠ DFE=∠ GLH=a right angle;
theref. Eu.VI. 8. BF^2=DBE= the rect. of the diff.

And thus it was anſwered by Mr. Swale.

XIV. QUESTION 62, *anſwered by Mr.* Lowry.

Conſ. In fig. 191, pl. 13, take CK = the given ſum of the ⊥ and biſecting line, and EQ a mean proportional between the other given ſum and difference; from E, to the indefinite right line IBD, drawn at right angles to CQ, apply FB=EQ; take BI = the given ſum of one ſide and its adjacent ſegment, and join IE. Draw EA to make the ∠ AEI = ∠ AIE, and ED to make the ∠ BED = ∠ BEA, ſo ſhall AED be the △ required.

Demon. EB biſects the ∠ AED, and EB+EC =CQ= the given ſum of the perpendicular and biſecting line.

Alſo, becauſe of the equal angles, AB=AI; therefore AE+AB=BI= the other given ſum by *Conſ.*

It

It remains now to prove that ED—BD is = to the given difference, to do which we have,

$(AE+AB)\cdot(ED-BD)=AED+AB\cdot ED-AE\cdot BD-ABD$,

and $AED-ABD=BE^2$;

theref. $(AE+AB)\cdot(ED-BD)=AB\cdot ED-AE\cdot BD+BE^2$:

but $AE : AB :: ED : BD$, ie. $AB\cdot ED=AE\cdot BD$;

therefore $(AE+AB)\cdot(ED-BD)=BE^2$,

and by *Conſ.* (AE+AB) × ed by the given diff. = BE^2;

therefore ED—BD = the given difference.

Q. E. D.

Cor. In any plane △, the line biſecting the vertical ∠ is a mean proportional between the ſum of one ſide and its adjacent ſegment, and the difference between the other ſide and its adjacent ſegment of the baſe, made by the ſaid biſecting line.

The ſame by Mr. Nicholſon.

Conſ. Make the rt. ∠'d △ BEC, (fig. 191, pl. 13.) ſuch, that the hypotenuſe BE may be a mean proportional between the given ſum of one ſide and its adjacent, and the given difference between the other ſide and its adjacent ſegment, and BE+EC = to the given ſum of the biſecting line and ⊥; take BO, in BC produced, = the given diff. join OE, and draw ED to meet BC produced in D, making the ∠ OED = ∠ DOE; laſtly, draw EA to make an ∠ with EB = to the ∠ BED, and let it meet BC produced in A, and ADE will be the △ required.

Demon. By *Conſ.* the ∠ AEB = ∠ DEB, the ∠ OED = ∠ DOE, EC ⊥ AD, and EC+EB = the given ſum of the ⊥ and biſecting line; therefore DE = DO; conſeq. DE—DB = DO—DB = BO, the given difference.

Now

Now Eu. VI. 3, $AE : ED :: AB : BD$, ie. $AE \cdot BD = AB \cdot ED$,

and $\quad AE \cdot ED = AB \cdot BD + BE^2$,

theref. $\quad (ED - BD) \cdot AE = (BD - ED) \cdot AB + BE^2$,

or, $\quad (ED - BD) \cdot AE - (BD + ED) \cdot AB = BE^2$:

but $(ED - BD) \cdot AE - (BD + ED) \cdot AB = (ED - BD) \cdot (AE + AB)$;

theref. by equality $(ED - BD) \cdot (AE + AB) = BE^2$.

Q. E. D.

The ſame anſwered by Mr. Swale.

Conſ. In BG, (fig. 191, pl. 13.) equal the given ſum of the ⊥ and biſecting line, take BE, = a mean proportional between the firſt given ſum and difference; with radius EG, and centre E, deſcribe the circle GCR, to which, through B, draw the tangent BC, and in BC produced, take BI, BO equal to the firſt given ſum and diff. reſpectively; join IE, OE, and biſect them in L, S, and demit the ⊥'s LA, SD, meeting BI in A and BC produced in D; then joining AE, DE; AED will be the required △.

Demon. EC being joined, is evidently ⊥ to AD. Since $IL = LE$, $OS = SE$, and LA, SD, ⊥ to IE, OE, the △'s IAE, ODE, are iſoſceles, therefore $IA = AE$, and $OD = DE$;

therefore $\quad BA + AE = BI =$ the given ſum,

and $\quad DE - DB = BO =$ the given difference.

Again, by a known Prop. $\quad AE : AB :: DE : DB$,

theref. comp. et div. $BI : AE - AB :: DE + DB : BO$:

but $(AE - AB) \cdot DE + DB) = AED - AEB = EB^2$;

therefore $\quad BI : BE :: BE : BO$, as by *Conſ.*

Alſo, ſince $EC = EG$, $EB + EC = BG =$ the given ſum, and EB biſects the ∠ AED, therefore,

Q. E. D.

XV.

XV. *QUESTION* 63, *answered by Mr.* Lowry.

Conſ. In fig. 192, pl. 13, take CO = the given diſtance between the vertex and the centre of the inſcribed circle, and produce it till the rect. OCh be = the given rectangle of the ſides; on hC as a diameter, let a circle be deſcribed, and apply therein Cf, Cg, each = half the given perimeter; demit the ⊥ OV; with the centres O and h, and diſtances OV, hf, let two circles be deſcribed, and let the right line AB be drawn to touch them both, and meeting Cg, Cf, in A, B, then ACB will be the △ required.

Demon. By *Conſ.* CO is = the given diſtance between the vertex and the centre of the inſcribed circle, and by a well known Prop. AC+CB+AB = 2Cf. Join OA, OB;

then CO : CB :: CE : CB+EB,
and CO : AC :: CE : AC+AE;
theref. CO^2 : ACB :: CE^2 : (CB+EB)·(AC+AE),
and by cor. to laſt qu. CE^2=(AC+AE)·(CB−EB);
therefore CO^2 : ACB :: CB—EB : CB+EB.
Again, AE : AC :: EB : CB;
theref. AE+AC : AC—AE :: EB+CB : CB—EB,
theref. per. et com. AE+AC+CB+EB : AC—AE+CB—EB :: CB+EB : CB—EB,
that is, 2Cf : 2CV :: CB+EB : CB—EB;
therefore Cf : CV :: ACB : CO^2;
but by *Conſ.* Cf : CV :: Ch : CO :: hCO : CO^2,
by equality ACB : CO^2 :: hCO : CO^2;
theref. ACB=hCO=the given rect. of the ſides.

Q. E. D.

Cor. In any plane triangle, the rectangle of the ſides is equal to the rectangle contained under the diſtances, between the vertex and the centre of the inſcribed circle, and the centre of a circle touching the baſe and the continuation of the ſides of the △.

Otherwiſe

Otherwiſe by Mr. Swale.

ANALYSIS.

Suppoſe ACB (fig. 192, pl. 13.) the △ that is to be conſtructed, and O the centre of the inſcribed circle. Biſect AB in H; through C, O, draw CI, meeting a ⊥ demited from H in I; join IA, IB; with the centre I and the diſtance IA=IB, deſcribe the circle AOBS, meeting CA again in L, and which will evidently paſs through O; produce CI to meet the circle again in S; to the points of contact of the inſcribed circle GVQ, draw the radii OG, OV, OQ, and upon AC demit the ⊥ IP.

Then it is known that CL=CB, and ACL=SCO, but a ACL is a given rect. and CO is a given line; therefore CS, CI, OI, are given.

Again, IO^2 ($=IA^2=IB^2$)=CIE, therefore IE, EC, are given. Alſo, AC+CB+AB (=2CP+AB=4CP−2CQ), the perimeter is given, and CP : CQ :: CI : CO, a given ratio; therefore CP, CQ, are given, and therefore the baſe AB, as well as the radius (OG=OQ=OV) of the inſcribed circle, is given, wherefore the ⊥ CR is given, and therefore the conſtruction of the triangle is evident.

XVI. QUESTION 64, *anſwered by Mr.* Swale.

ANALYSIS.

Let AFE (fig. 193, pl. 13.) be the △ to be conſtructed. Demit the ⊥ FD, and make DC = DE, and biſect AC in B; join FC, FB, and upon FB demit the ⊥ DL.

Since AC=2BC, and AF^2+FE^2($=AF^2+FC^2$ $=2BF^2+2BC^2$) are given, BF will be given.

Again,

Again, the rect. $\frac{1}{2}$ AE·FD=BDF=BF·LD is given, therefore DL or the point L is given.

The Composition, by Mr. Lowry.

Const. Take BF^2 = the difference between half the given sum of the squares and the square of half the difference of the segments of the base, and on BF constitute the rectangle BFST to contain the given area, and meeting a semicircle described upon BF, in D; draw BD, and on it continued, take AB, BC, each equal to half the given difference of the segments of the base, and make DE = DC, and join EF; AFE is the △ required.

Demon. AD—DE = AD — DC = AC = the given diff. and $AF^2+EF^2=AF^2+FC^2=2AB^2+2BF^2$: but $2BF^2$ = the given sum of the squares—$2AB^2$; theref. AF^2+EF^2=the given sum of the sqs. Again, △ AFE=2 △ BFD= rect. BFST= the given area.

Q. E. D.

An elegant construction was also received from Mr. Richard Nicholson.

XVII. *QUESTION* 65, *answered by Mr.* Louis Hill.

Const. In any line take FM (fig. 196, pl. 14.) = the given sum of the ⊥ and diff. of the segments of the base, and perpendicular thereto, take FH=$\frac{1}{2}$ FM; join MH, and draw FQ to meet it in Q, and making the ∠QFH = to half the given diff. of the ∠'s at the base; on MQ let a semicircle be described, and draw HIK ⊥ to MQ, and let it meet the circle in I; take HK a 4th proportional to FH, HI, and S, S, being = to half the given sum of the sides. Draw KL ∥ to MQ, meeting the circle

circle in L, and on MQ drop the $\perp$ LD, and draw DEC $\parallel$ to FH, meeting FM in E, and take thereon EC $=$ DE; draw DG $\parallel$ to FQ, meeting FM in Q, and through the three points G, C D, defcribe a circle, cutting FH produced in B and A; join CA, BC, and ACB will be the $\triangle$ that was to be conftructed.

Demon. On AB demit the $\perp$ CP, join DB, and continue QF to meet DC produced, in N.

Then BP—AP$=$CD, as is well known, and FM $=$ 2FH; theref. ME$=$2ED$=$CD, and theref. CP$+$CD$=$FM$=$ the given fum of the $\perp$ and diff. of the fegments of the bafe.

Again, by $\parallel$'s $\angle$ DGE $=\angle$ NFG, and theref. $\angle$ GDE$=\angle$QFH; confeq. $\angle$ BAC— $\angle$ABC$=$ $\angle$CBD$=$2$\angle$CDG$=$2$\angle$QFH$=$ the given diff. of the angles at the bafe.

By fim. $\triangle$'s $FH : HQ :: ND : DQ$,
and $FH : HM :: DE : DM$,
therefore, $FH^2 : MHQ :: EDN : MDQ$:
but, by *Conf.* $FH : S :: HI : HK$,
that is, $FH^2 : S^2 :: HI^2 : HK^2$,
and $HI^2 = MHQ$, $HK^2 = DL^2 = MDQ$;
theref. by equality, $EDN = S^2$,
but, Simp. Trig. Prop. 18. $\overline{\tfrac{1}{2}(AC+CB)}^2 = EDN$;
theref. $\overline{\tfrac{1}{2}(AC+CB)}^2 = S^2$, and confequently AC$+CB=$ the given fum of the fides.

Q. E. D.

The conftructions received from Meffrs. Lowry, Nicholfon, *and* Swale, *being fo very little different from the one given above, it is unneceffary to exhibit them, even if we had room.*

XVIII. QUESTION 66, *anfwered by Mr.* Lowry.

Conf. Fig. 196, pl. 14, draw the right lines BCR, BS, to contain the given diff. of the $\angle$'s at

 the

the base; take BR, BS. each = to half the given sum, and erect RO, SO, perpendicular to them intersecting each other in O; draw OB, and divide it in W. so, that the rect. OBW may be = the given rectangle of the sides; with the centres O, W, describe two circles to touch the right lines BR, BS, and draw CD to touch both the circles, and intersect BR, BS, in C, D; through the points C, D, B, describe a circle, and draw BA parallel to DC, meeting the circle in A; join CA, and ACB, will be the $\triangle$ required.

Demon. $\angle BAC \angle - ABC \angle = CBD =$ the given diff. of the $\angle$'s at the base, and $AC + CB + CD = CB + BD + CD = BR + BS$, by a well-known property, = the given sum of the sides and diff. of the segments of the base. Again, $AC \cdot CB = CB \cdot BD = OB \cdot BW$, by *Cor.* to Qu. 63, = the given rectangle by construction. *Q. E. D.*

Constructions were also received from Messrs. Hill, Nicholson, *and* Swale.

XIX. QUESTION 67, *answered by the Rev. Mr.* L. Evans.

In Simpson's Fluxions, page 76, we have $x - \dfrac{x + n^2a^2x^{2n-1}}{n-1}$, an expression for the sum of the abscissa of the evolute, and the radius of curvature at the vertex of a parabola of any kind; but when $x = 0$, and $n = \frac{1}{2}$, the expression becomes $-\dfrac{\frac{1}{4}a^2 \times 0^0}{-\frac{1}{2}} = \dfrac{a^2}{2}$, the value of the radius of curvature of the conical parabola at the vertex.

But this mode of expression ($0^0 = 1$) can only be properly understood and applied by those mathematicians who are acquainted with the use of this notation: it is not intended for the practice of such

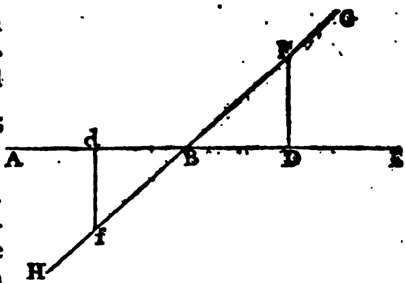
as SEARCH, and NO CONJURER, who have ſhewn themſelves alike incapable of aſcertaining its meaning, or diſproving its application. Theſe gentlemen have evidently miſtaken the expreſſion 0^0, for $\frac{0}{0}$, as is plain from what has been aſſerted in the Monthly Magazine, where it is ſaid (vol. 1. p. 126), "I England our mathematicians are content with making nothing divided by nothing equal to unity." From this aſſertion we may reaſonably conclude, that the writer of it was as little acquainted with the principles of the method he attempted to controvert, as the "*farmer's pigs*," which he ſuppoſes "*nobody* let into the garden."

As this ſolution may fall into the hands of the gentlemen alluded to, who, however expert they may be in weaving metaphors, or detecting imaginary errors, are certainly not quite infallible in forming mathematical deductions; I ſhall prove that the reſult of the above equation will be the ſame $\left(\frac{a^2}{2}\right)$ whether we uſe the expreſſion 0^0, or its equal 1.

Multiply the equation by x, and we have $x^2 - \frac{\overline{x^2 + n^2 a^2} \times x^{2n}}{n-1} = xy$; make $n = \frac{1}{2}$, and it will be $x^2 - \frac{x^2 * \frac{1}{4}a^2 \times x}{-\frac{1}{2}} = xy$, or $y = x - \frac{\overline{x + \frac{1}{4}a^2} \times 1}{-\frac{1}{2}}$, equal, when $x = 0$, to $-\frac{\frac{1}{4}a^2 \times 1}{-\frac{1}{2}} = \frac{a^2}{2}$ as before.

Hence, then, it appears, that the mode of expreſſion, againſt which theſe ſagacious reaſoners have raiſed ſuch a clamour, is conſiſtent in itſelf, and rational in its application; and it may be obſerved, that real mathematicians are too well acquainted with its uſe to be prevailed upon to diſcard it, becauſe every "dabbler" may not at firſt ſight be able to apprehend its utility.

If the equation in the 20th question, No. 2, had been $\frac{bx - ab}{c} = y$; the locus would have been the rt. line GH cutting the right line AE in the given



angle ABH, where AB $= a$, AD $= x$, FD $= y$; and BD to BF in the given ratio of c to b; or when bx is less than ab; Bd to df in the given ratio of c to b.

But the author* of a book lately published, has pronounced DF (y) impossible when bx is less than ab: his reason for this it is not difficult to perceive, as he has, in several parts of the work, given abundant reason for presuming that he knows of no other use of the negative sign than that of subtracting a less quantity from a greater.

It was also answered by Mr. Swale.

XX. *QUESTION* 68, *answered by Mr.* Gompertz.

1. Let the radii of the arcs BC, BA (fig. 194, pl. 13.) be denoted by r and s, the constant distance AC by b, the versed-sines, corresponding to arcs similar to BC, BA, to radius unity, by x and v, and the arcs themselves by A and B. Then the arc BC $= r$ A, and the arc BA $= s$B; also, DC $= rx$, and AD $= sv$; theref. $rx + sv = b$, and $2r^2x - r^2x^2 =$ BD$^2 = 2s^2v - s^2v^2 = 2s\cdot(b - rx) - (b - rx)^2$; theref. $x = (2ab - 2rb - b^2) \div r\cdot(2a - 2b)$, and $v = (2ab - 2sb - b^2) \div s\cdot(2a - 2b)$, a being $= r + s$; let us now put $(2ab - b^2) \div (2a - 2b) = c$, and $2b \div (2a - 2b) = d$,

* May we not justly suppose that this author, is the same person who stiles himself "No Conjurer," in the Monthly Magazine.

and

and we shall have $x = c \div r - d$, and $v = c \div s - d$; theref. $\dot{r} = -\dot{s}$, $\dot{x} = -c\dot{r} \div r^2$, and $\dot{v} = c\dot{r} \div s^2$, $\dot{A} = -c\dot{r} \div r^2\sqrt{(2x-x^2)}$, $\dot{B} = c\dot{r} \div s^2\sqrt{(2v-v^2)}$. But $rA + sB$ is an extreme value, that is, $\dot{r}A + r\dot{A} + \dot{s}B + s\dot{B} = 0$; this, by substitution and reduction becomes, $A - c \div r^2\sqrt{(2x-x^2)} + c \div s^2\sqrt{(2v-v^2)} - B = 0$; but, from the equations above, it is easy to exterminate any three of the four quantities r, s, x, v, and by this means all the four quantities will become known.

2. The notation remaining as before, only A and B, in this case, denoting the areas of segments similar to BCD, BAD, to radius unity, we shall have the area $DBC = r^2A$, the area $BAD = s^2B$, $\dot{A} = \dot{x}\sqrt{(2x-x^2)} = -c\dot{r}\sqrt{(2x-x^2)} \div r^2$, $\dot{B} = c\dot{r}\sqrt{(2v-v^2)} \div s^2$, and $\dot{A}r^2 + 2r\dot{r}A + \dot{B}s^2 + 2s\dot{s}B = 0$, and from this equation, as in the first case, the values of r, s, x, and v, may be found.

3. Let p denote the latus rectum of the parabola, the abscissa $AD = pv$, and A the arc of a parabola similar to AB, whose latus rectum is unity, B the arc of a circle similar to BC, to radius unity, and C the area of a segment similar to BCD, to radius unity. Hence $DC = rx$, $pv + rx = a$, $p^2v = 2r^2x - r^2x^2$, $\frac{2}{3}p^2v^{\frac{3}{2}} + r^2C = b$, the given area; whence $p = (a-rx) \div v$, $(a-rx)^2 \div v = 2r^2x - r^2x^2$, $\frac{2}{3}(a-rx)^2 \div v^{\frac{1}{2}} + r^2C = b$; therefore $\frac{4}{9}(a-rx)^4 \div (b-r^2C)^2 = v = (a-rx)^2 \div (2r^2x - r^2x^2)$, and therefore $\frac{2}{3}(ar - r^2x) = (b - r^2C) \div \sqrt{(2x-x^2)}$.

From this last equation, r may be found in terms of x, C, and known quantities, and from thence v and p. Again, since $\dot{C} = \dot{x}\sqrt{(2x-x^2)}$, we may easily find $\dot{r}$, $\dot{v}$, and $\dot{p}$, in terms of $\dot{x}$ multiplied by values expressed by x, C, and known quantities,

 which

which values let us represent by R, P, and Q, that is, $\dot{r}=R\dot{x}$, $\dot{v}=P\dot{x}$, $\dot{p}=Q\dot{x}$. But $AB=pA$, $BC=rB$, and $\dot{AB}+\dot{BC}=0$; therefore $Q\cdot A+P\cdot p\sqrt{1+1\div 4v}+R\cdot B+r\div\sqrt{(2x-x^2)}=0$, a finite equation, in terms of circular arcs and logarithms.

Q. E. I.

Scholium. In the same way, as in the last case, may the Prize Question in Ladies' Diary, 1741, be solved. Thus AB being found, as in the Diary, $=44\cdot945$, &c. let p represent the latus rectum of the parabola sought, $AS=py$, $VS=py^2$, and A the arc of a parabola whose ordinate is y, and latus rectum unity. Hence $AV=pA=44\cdot945=a$, and $p^2\ y^3$, or $y^3\div A^2$, a maximum, which being fluxed, and $\dot{y}\sqrt{1+4y^2}$ substituted for $\dot{A}$, its equal, we shall have, by reduction, $3A=2y\sqrt{1+4y^2}$, that is, because $A=\frac{1}{2}y\sqrt{1+4y^2}+\frac{1}{4}$ H. L. $(2y+\sqrt{1+4y^2})$, H. L. $(2y+\sqrt{1+4y^2})=\frac{2}{3}y\sqrt{1+4y^2}$, and therefore $y=\cdot9731$, $A=1\cdot42$, &c. $p=31\cdot65$, &c. $py=AS=30\cdot8$, &c.

Mr. Swale *answered it.*

ARTICLE LVII.

We have been favoured with the following solution, from Caput Mortuum,* *to Question* 46, *proposed in No. II. to which, we had not before received any that was satisfactory.*

THIS question, in its present form, is indeterminate, because the angle depends on the direction of the tangent. But that being given, an an-

* We intreat the favour of this able Gentleman's future correspondence, whose communications shall have all due deference and regard paid to them.

fwer

ſwer may be derived from what follows. Let CE, CP, (fig. 195, pl. 13.) be the equatorial and polar ſemi-axes of the ſpheroid; EP, AP, two meridians paſſing through the two given points D, R; and *dp*, *rp*, the two correſponding meridians on a ſphere whoſe centre is C. Draw the verticals RQ, DB; and the radii *r*C, *d*C, parallel to RQ, DB, reſpectively. Then will the latitudes of the points R, *r*; D, *d*, and diff. of longitude, be reſpectively the ſame on both figures. And becauſe *r*C is parallel to RQ, and *d*C to DB, the planes of the horizons at R, and D, will be parallel to the planes of the horizons (or planes touching the ſphere) at *r* and *d*. Therefore, if RN be a tangent on the ſpheroid, making a given angle (NRP) with the meridian RP, and *rn* a tangent on the ſphere, making the ſame angle (*nrp*) with *rp*, thoſe tangents will be parallel to each other, and when produced, will meet the oppoſite planes, or horizons at D, and *d*, in equal angles, becauſe parallel lines cut parallel planes in equal angles. *Hence, the computed angle on a ſphere will anſwer the conditions on a ſpheroid.*

ARTICLE LVIII.

MATHEMATICAL QUESTIONS,

(To be anſwered in Number VII.)

I. QUESTION 89, *by Mr.* Newton Boſworth.

THE ſide of the greateſt equilateral triangle that can be inſcribed in the generating circle of a cycloid=16.—It is required from hence to determine the content of the ſolid formed by a rotation of the cycloid about the tangent parallel to its axis?

II. QUESTION 90, *by a* Gardener.

Having lately purchaſed a rectangular garden, whoſe length is 100, and breadth 80 yards, I wiſh

to make a gravel walk therein, of equal width along the middle lengthwise, and across one end of the breadth, so as to take up just 1-20th part of the garden. But being unskilled in figures, I shall be obliged to any of the ingenious correspondents to the Repository to inform me, what the width of the walk must be?

III. QUESTION 91, *by Mr.* John Johnston.

If an egg, in the form of a spheroid, whose longer axis is 2 inches, and the lesser axis 1 inch, be let fall into a conical glass, filled with water to within half an inch of the top, whose diameter is 2, and altitude 4 inches; it is required to determine how much water will run over?

IV. QUESTION 92, *by Mr.* John Harris.

In the pleasant vale of Towy, near Caermarthen, lies a fertile semicircular meadow, containing 20 acres, which is the equal property of three neighbouring gentlemen, who have agreed to divide it into three equal parts, by fences to be made from a watering place in the diameter, at 6 chains distance from the centre. Now the surveyor they have employed for the purpose of dividing it, not being able to effect it by means of scale and compass, and being, like most of his profession, unskilled in calculation, humbly solicits some gentleman, more ingenious than himself, to inform him what angles the fences must make with the diameter, and what it will come to at *6d.* per yard fencing?

V. QUESTION 93, *by Mr.* Harris.

A friend of mine has a small field in the form of the segment of a circle containing 9 acres, and worth 40*l.* per acre, the versed-sine of which is 2 chains. Now, in its present form, he finds it very inconvenient for cultivation, and has, therefore, agreed with the

the owner of the adjoining field, on the curve ſide, to have it laid out in a ſquare form, beginning at one corner of the ſegment. But his neighbour's land, for want of proper cultivation, and other circumſtances, being worth only 30*l*. per acre, he requeſts ſome ingenious gentleman, whoſe labours ſo eminently adorn the Repoſitory, to favour him with the plan and content of the intended diviſion, ſo that neither party may be injured.

VI. QUESTION 94, *by Mr.* Bulmer.

In the lat. of 51° 32′ N. I found the ſum of the ſun's altitude and declination $= 69° \ 48'$, and the ſum of his azimuth from the north and the hour angle $= 156° \ 53'$. *Quere* the time when the obſervation was made, the declination being north?

VII. QUESTION 95, *by the Rev. Mr.* L. Evans.

If the tranſverſe axis of a given ellipſis revolve around one of the foci, ſo, that one end may always be in the periphery; it is required to find the equation, &c. of the curve which is the *locus* of the other end of the axis?

VIII. QUESTION 96, *by Mr.* Ralph Simpſon, *taken from Dr.* Hutton's *Select Exerciſes.*

A very large veſſel, of 10 feet high (no matter of what ſhape), being kept conſtantly full of water, by a large ſupplying cock at top; if nine ſmall circular holes, each $\frac{1}{5}$ of an inch diameter, be opened in its perpendicular ſide at every foot of the depth; it is required to determine the ſeveral distances to which they will ſpout on the horizontal plane of the baſe, and the quantity of water diſcharged by all of them in ten minutes?

IX. QUESTION 97, *by Mr.* J. H. Swale.

There are two right lines BP, PG, given by poſition, and meeting in P; in which are two given points

points B, D, and in a direct line between them a point A. From D, a body d moves in the line DG, towards G, with a given celerity. Now suppose a body a remain stationary at A; it is required to determine with what celerity a body b must move in the line BP, so that the three bodies d, a, b, may be always in a right line, so long as b continues below A with respect to P?

X. QUESTION 98, *by Mr.* Thornoby.

Suppose a globe resting between two inclined planes, which cut one another at right angles; it is required to find what proportion of the pressure each plane sustains, the length and perpendicular altitude of each plane being given?

XI. QUESTION 99, *by Mr.* Thornoby.

Three posts, A = 5, B = 4·5, C = 4 feet long, are set erect upon the horizon, at the distances AB = 9·5, BC = 9, CA = 8·5 feet; three rafters, AD = 7·15, BD = 6·65 CD = 6·15 feet long, are placed on these, and unite at the top D; it is required to find how far the point, perpendicularly under D, in the horizontal plane, is from the bottom of each post?

XII. QUESTION 100, *by* Astronomicus.

To find the declination of that star which changes its declination the greatest quantity possible in passing over the interval contained between two given hour circles, in a given latitude.

XIII. QUESTION 101, *by Mr.* Richard Nicholson.

Given the base and ⊥ of a plane △ to construct it when the rectangle of the sides is equal to twice the rectangle of the segments of the base made by the line bisecting the vertical angle.

XIV.

XIV. QUESTION 102, *by Mr.* M. A. Harriſon.

Given the rectangle of the ſegments of the baſe made by the point of contact of the inſcribed circle, the ⊥, and the difference of the ſegments of the baſe made thereby to conſtruct the plane △.

XV. QUESTION 103, *by Mr.* Swale.

Given the vertical angle, the ſegment of the line biſecting it, made by perpendiculars from the extremities of the baſe, and the difference of the ſides to conſtruct the △.

XVI. QUESTION 104, *by Mr.* Swale.

Given the ſum of the ſquares of the ſides and the difference of the ſegments of the baſe made by the ⊥, to conſtruct the △ when the ſolid under the ſquare of the ⊥ and baſe is a maximum.

XVII. QUESTION 105, *by Mr.* W. Peacock.

Given the baſe, the difference of the ſides, and the ſegment intercepted between the vertex and a ⊥ from one of the ∠'s at the baſe upon the oppoſite ſide, to conſtruct the plane △.

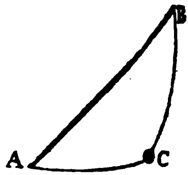
XVIII. QUESTION 106, *by Mr.* Louis Hill.

Given the difference of the ſides, the ⊥, and the ratio of the ſegments of the baſe made thereby to conſtruct the triangle.

XIX. QUESTION 107, *by Mr.* John Lowry.

Given the vertical angle, the area, and the diſtance between the centre of the inſcribed circle and the centre of a circle touching the baſe and the continuation of the two ſides to conſtruct the plane △.

XX. *QUESTION* 108, *by Mr.* Lowry.



Given the length of the rod AB, and the length of the ſtring ACB, on which ſlides freely the ring of heavy metal C; to find the nature of the curve deſcribed by the ring while the rod revolves with an uniform velocity about the angular point A.

XXI. *PRIZE QUESTION* 109, *by Mr.* Lowry.

Given the perpendicular of a plane triangle, to conſtruct it when the baſe paſſes through a given point within a given circle, and terminates in the circumference; and the ratio of the ſquares of the two ſides is the ſame as the ratio of the ſegments of the baſe made by the given point.

ARTICLE LIX.

Demonſtrations to Dr. Stewart's *Propoſitions propoſed in* ARTICLE XXXII.

PROP. XIX. THEO. XVI. *Fig.* 133, *Plate* 9.

Demonſtrated by Dr. Small.

LET there be any number, *m*, of ſtraight lines AB, AC, AD. AE, &c. given by poſition, interſecting one another in the point A, and let *a*, *b*, *c*, *d*, &c. be given magnitudes, as many in number as there are lines given by poſition, two ſtraight lines AX, AY, may be found which will be given by poſition, ſuch, that if from any point F there be drawn FB, FC, FD, FE, &c. perpendicular to AB, AC, AD, AE, &c. and FX, FY, perpendicular to AX, AY,

FB²

$$FB^2+\frac{b}{a}FC^2+\frac{c}{a}FD^2+\frac{d}{a}FE^2\ \&c. = \frac{a+b+c+d}{2a}(FX^2+FY^2).$$

Let $m=4$. Let G be the centre of the circle which paſſes through the points A, B, C, D, E, and F; and let H be the centre of gravity of weights proportional to the magnitudes a, b, c, d, placed at the points B, C, D, and E. Join GH; and let XY, at right angles to GH in H, meet the circumference of the circle ABDF in X and Y: AX, AY are the lines required to be found.

For it may be ſhewn, juſt as in Theo. 12. by means of a lemma ſimilar to the 3d, that AX and AY make given angles with AB, and are therefore given in poſition. But by Theo. 7.

$$GB^2+\frac{b}{a}GC^2+\frac{c}{a}GD^2+\frac{d}{a}GE^2\&c.\frac{a+b+c+d}{a}GX^2=$$

$$HB^2+\frac{b}{a}HC^2+\frac{c}{a}HD^2+\frac{d}{a}HE^2=\frac{a+b+c+d}{a}GH^2.$$

$$\text{Now}\ \frac{a+b+c+d}{a}GX^2=\frac{a+b+c+d}{2a}(GX^2+GY^2)=$$

$\frac{a+b+c+d}{a}(GH^2+HX^2)$, by Prop. 1. Therefore,

$$HB^2+\frac{b}{a}HC^2+\frac{c}{a}HD^2+\frac{d}{a}HE^2=\frac{a+b+c+d}{a}HX^2.$$

Again by Theo. 7. $FB^2+\frac{b}{a}FC^2+\frac{c}{a}FD^2+\frac{d}{a}FE^2=$

$HB^2+\frac{b}{a}HC^2+\frac{c}{a}HD^2+\frac{d}{a}HE^2+\frac{a+b+c+d}{a}HF^2$; therefore,

$$FB^2+\frac{b}{a}FC^2+\frac{c}{a}FD^2+\frac{d}{a}FE^2=\frac{a+b+c+d}{a}(HX^2+HF^2),$$

or, ſince $HX^2+HF^2=\frac{1}{2}(FX^2+FY^2)$,

$$FB^2+\frac{b}{a}FC^2+\frac{c}{a}FD^2+\frac{d}{a}FE^2=\frac{a+b+c+d}{2a}(FX^2+FY^2).$$

The same Demonstrated by Mr. Lowry, Fig. 159, Pl. 11.

Let there be any number of right lines AB, AC, AD, AE, &c. given by poſition, interſecting each other in the point A, and let *a*, *b*, *c*, *d*, &c. be given magnitudes, as many in number as there are right lines given by poſition; two right lines AY, AZ may be found which will be given by poſition, ſuch, that if from any point X, there be drawn the perpendiculars XB, XC, XD, XE, &c. to the right lines AB, AC, AD, AE, &c. given by poſition, and likewiſe XY, XZ perpendiculars to AY, AZ, the two lines found; the ſquare of XB, together with the ſpace to which the ſquare of XC has the ſame ratio that *a* has to *b*, together with the ſpace to which the ſquare of XD has the ſame ratio that *a* has to *c*, and ſo on, will be equal to the ſpace to which the ſum of the ſquares of XY, XZ, has the ſame ratio that twice *a* has to the ſum of *a*, *b*, *c*, &c.

Join AX, and biſect it in Q; with the centre Q and diſtance QX, or QA, deſcribe a circle interſecting the given lines in B, C, D, E, &c. Find the point V, as in Prop. X. for the points B, C, D, E, &c. and the given magnitudes *a*, *b*, *c*, *d*, &c. Join VQ, and at right angles thereto, draw YVQ, meeting the circle in Y and Z; join AY, AZ, and they will be the two right lines that were to be found.

Join QB, QC, QD, QE, &c. VB, VC, VD, VE, &c. and VX, QY, QZ; XB, XC, XD, XE, &c. and XY, XZ, being joined, will be perpendicular to AB, AC, AD, AE, &c. and AY, AZ, reſpectively.

By Prop. X. the ſquare of QB, together with the ſpace to which the ſquare of QC has the ſame ratio that *a* has to *b*, together with the ſpace to which the ſquare of QD has the ſame ratio that *a* has to *c*, together with the ſpace to which the ſquare of QE has the ſame ratio that *a* has to *d*, and ſo on, that

is,

is, the ſpace to which the ſquare of the ſemidiameter QY of the circle has the ſame ratio that a has to the ſum of a, b, c, d, &c. is equal to the ſquare of VB, together with the ſpace to which the ſquare of VC has the ſame ratio that a has to b, together with the ſpace to which the ſquare of VD has the ſame ratio that a has to c, together with the ſpace to which the ſquare of VE has the ſame ratio that a has to d, and ſo on, together with the ſpace to which the ſquare of VQ has the ſame ratio that a has to the ſum of a, b, c, d, &c. But the ſquare of the ſemidiameter QY is equal to the ſum of the ſquares of VY, VQ. Therefore the ſpace to which the ſum of the ſquares of VY, VQ, has the ſame ratio that a has to the ſum of a, b, c, d, &c. is equal to the ſquare of VB, together with the ſpace to which the ſquare of VC has the ſame ratio that a has to b, together with the ſpace to which the ſquare of VD has the ſame ratio that a has to c, together with the ſpace to which the ſquare of VE has the ſame ratio that a has to d, and ſo on, together with the ſpace to which the ſquare of VQ has the ſame ratio that a has to the ſum of a, b, c, d, &c. that is, the ſpace to which the ſquare of VY has the ſame ratio that a has to the ſum of a, b, c, d, &c. is equal to the ſquare of VB, together with the ſpace to which the ſquare of VC has the ſame ratio that a has to b, together with the ſpace to which the ſquare of VD has the ſame ratio that a has to c, together with the ſpace to which the ſquare of VE has the ſame ratio that a has to d, and ſo on.

Again, by Prop. X. the ſquare of XB, together with the ſpace to which the ſquare of XC has the ſame ratio that a has to b, together with the ſpace to which the ſquare of XD has the ſame ratio that a has to c, together with the ſpace to which the ſquare of XE has the ſame ratio that a has to d, and ſo on, is equal to the ſquare of VB, together with the ſpace to which the ſquare of VC has the ſame ratio that

a has to *b*, together with the ſpace to which the ſquare of VD has the ſame ratio that *a* has to *c*, together with the ſpace to which the ſquare of VE has the ſame ratio that *a* has to *d*, and ſo on, together with the ſpace to which the ſquare of VX has the ſame ratio that *a* has to the ſum of *a*, *b*, *c*, *d*, &c. Therefore the ſquare of XB, together with the ſpace to which the ſquare of XC has the ſame ratio that *a* has to *b*, together with the ſpace to which the ſquare of XD has the ſame ratio that *a* has to *c*, together with the ſpace to which the ſquare of XE has the ſame ratio that *a* has to *d*, and ſo on, is equal to the ſpace to which the ſum of the ſquares of VX, VY, has the ſame ratio that *a* has to the ſum of *a*, *b*, *c*, *d*, &c. But YZ is biſected in V; therefore (by Prop. II. *Cor.*) the ſum of the ſquares of XY, XZ, is equal to twice the ſum of the ſquares of VX, VY. Therefore the ſquare of XB, together with the ſpace to which the ſquare of XC, has the ſame ratio that *a* has to *b*, together with the ſpace to which the ſquare of XD has the ſame ratio that *a* has to *c*, together with the ſpace to which the ſquare of XE has the ſame ratio that *a* has to *d*, and ſo on, is equal to the ſpace to which the ſum of the ſquares of XY, XZ, has the ſame ratio that twice *a* has to the ſum of *a*, *b*, *c*, *d*, &c.

The ſame Demonſtrated by Mr. Swale, *Fig.*197, *Pl.*14.

Let there be any number of right lines AB, AC, &c. given by poſition, interſecting each other in the point A, and let *a*, *b*, &c. be given magnitudes, as many in number as there are right lines AB, AC, &c. given by poſition; two right lines QR, ST, may be found that will be given by poſition, ſuch, that if from any point P there be drawn the perpendiculars PB, PC, &c. to the right lines AB, AC, &c. given by poſition, and likewiſe there be drawn PF, PG, perpendiculars to QR, ST, the two lines found,

found, the ſquare of PB, together with the ſpace to which the ſquare of PC has the ſame ratio that a has to b, and ſo on, will be equal to the ſpace to which the ſum of the ſquares of PF, PG, has the ſame ratio that twice a has to the ſum of a, b, c, d, &c.

Suppoſe two lines AB, AC, to be given by poſition, and interſecting each other in the point A.

From any point P, demit the perpendiculars PB, PC, and join BC; take BD to DC as b to a, and join PD; make DE perpendicular to BC, and let it meet a ſemicircle deſcribed thereon in E; in PD, take PI = DE; make DF, IG, perpendiculars to PD, and equal to PD, PI, reſpectively, and join PF, PG. Perpendicular to the lines PF, PG, given by poſition, draw QR, ST, given alſo by poſition, and they will be two ſuch lines as were required to be found.

For, a times the ſquare of PB, together with b times the ſquare of PC, is equal to the multiple of the ſquare of PD by the ſum of a, b, together with the multiple of the rectangle BDC by the ſum of a, b, that is, equal to the multiple of the ſum of the ſquares of PD, DE (PI) by the ſum of a, b. Therefore, the ſquare of PB, together with the ſpace to which the ſquare of PC has the ſame ratio that a, has to b, is equal to the ſpace to which the ſum of the ſquares of PD, PI, has the ſame ratio that a has to the ſum of a, b; that is, equal to the ſpace to which twice the ſum of the ſquares of PD, PI, has the ſame ratio that twice a, has to the ſum of a, b. But (by conſtruction) twice the ſum of the ſquares of PD, PI, is equal to the ſum of the ſquares of PF, PG. Therefore the ſquare of PB, together with the ſpace to which the ſquare of PC has the ſame ratio that a has to b, is equal to the ſpace to which the ſum of the ſquares of PF, PG, has the ſame ratio that twice a has to the ſum of a, b.

Note. The extenſion of the above method to any

number of lines at pleafure, is fufficiently manifeft to an intelligent reader.

Cor. Let there be any number of right lines given by pofition interfecting each other in a point; two right lines may be found that will be given by pofition, fuch, that if from any point there be drawn right lines in given angles to all the right lines given by pofition, and likewife there be drawn perpendiculars to the two lines found, the fum of the fquares of the lines drawn in given angles to the right lines given by pofition, will be equal to the fpace to which the fum of the fquares of the perpendiculars drawn to the two lines found has a given ratio.

PROP. XX. THEO. XVII. Fig. 161, Plate 11.

Demonftrated by Mr. Lowry.

Let there be any number of right lines AB, BC, CD, DA, &c. that are neither all parallel nor interfecting each other in one point, and let *a*, *b*, *c*, &c. be given magnitudes as many in number as there are right lines given by pofition; two right lines PY, PZ, may be found that will be given by pofition, fuch, that if from any point X there be drawn the perpendiculars XE, XF, XG, XH, &c. to the right lines AB, BC, CD, DA, &c. given by pofition, and XY, XZ perpendiculars to PY, PZ, the two right lines found, the fquare of XE, together with the fpace to which the fquare of XF has the fame ratio that *a* has to *b*, together with the fpace to which the fquare of XG has the fame ratio that *a* has to *c*, and fo on, will be equal to the fpace to which the fum of the fquares of XY, XZ, has the fame ratio that twice *a* has to the fum of *a*, *b*, *c*, &c. together with a given fpace.

From the point X draw XE, XF, XG, XH, &c. perpendiculars to the right lines AB, BC, CD, DA, &c. given by pofition, and let the point Q be found

as in Prop. X. for the points E, F, G, H, &c. and the given magnitudes a, b, c, &c. From the interſection A, of two of the right lines given by poſition, draw AI, AK, &c. parallel to the reſt of the lines BC, CD, &c. given by poſition. Then, by Prop. XIX. let two right lines AR, AL, be found for the point X, the right lines AB, AK, AI, AD, &c. and the given magnitudes a, b, c, &c. Draw XR, XL, perpendiculars to AR, AL; and through Q draw SQT perpendicular to XSL; take QT equal to QS, and draw TY parallel to SX, meeting XR, produced if neceſſary, in Y; draw YQ, and let it meet XL in Z; then, if YP, ZP, interſecting each other in P, be drawn parallel to AR, AL, they will be two ſuch lines as were required.

It is ſhewe in the ſame manner as in Prop. XVI. (by referring to Prop. X. and XIX. inſtead of Prop. IX. and XV.) that the ſquare of XE, together with the ſpace to which the ſquare of XF has the ſame ratio that a has to b, together with the ſpace to which the ſquare of XG has the ſame ratio that a has to c, and ſo on, is equal to the ſquare of PM, together with the ſpace to which the ſquare of PN has the ſame ratio that a has to b, together with the ſpace to which the ſquare of PO has the ſame ratio that a has to c, and ſo on, is equal to the ſpace to which the ſum of the ſquares of XY, XZ, has the ſame ratio that twice a has to the ſum of a, b, c, &c. But the point P has been found in the conſtruction; therefore the right lines PM, PN, PO, &c. are given. And therefore the ſquare of PM, together with the ſpace to which the ſquare of PN, has the ſame ratio that a has to b, together with the ſpace to which the ſquare of PO has the ſame ratio that a has to c, and ſo on, is equal to a given ſpace; therefore the ſquare of XE, together with the ſpace to which the ſquare of XF has the ſame ratio that a has to b, together with the ſpace to which the ſquare of XG has the

ſame

ſame ratio that a has to c, and ſo on, is equal to the ſpace to which the ſum of the ſquares of XY, XZ, has the ſame ratio that twice a has to the ſum of a, b, c, &c. together with a given ſpace.

The ſame Demonſtrated by Mr. Swale.

Let there be any number of right lines BG, CH, DI, &c. (fig. 198, pl. 14,) given by poſition, that are neither all parallel nor interſecting each other in one point, and let a, b, c, &c. be given magnitudes, as many in number as there are right lines BG, CH, DI, &c. given by poſition; two right lines TP, RL, may be found that will be given by poſition, ſuch, that if from any point A there be drawn AB, AC, AD, &c. perpendiculars to the right lines BG, CH, DI, &c. given by poſition, and likewiſe there be drawn AP, AL, perpendiculars to TP, RL, the two right lines found, the ſquare of AB, together with the ſpace to which the ſquare of AC, has the ſame ratio that a has to b, together with the ſpace to which the ſquare of AD, has the ſame ratio that a has to c, and ſo on, will be equal to the ſpace to which the ſum of the ſquares of AP, AL, has the ſame ratio that twice a has to the ſum of a, b, c, &c. together with a given ſpace.

Let there be three right lines BG, CH, DI, given by poſition, and from any point A, let fall the perpendiculars AB, AC, AD. Join BC, and take BE to EC as b to a; join AE, DE, and take EF to FD as the ſum of a, b, to c; join AF, and make FK perpendicular to DE, meeting a ſemicircle deſcribed thereon in K: in AE take AN=FK; draw FL, NP, perpendiculars and equal to AF, AN, reſpectively; join AL, AP, and draw LR, PT, perpendicular thereto, and they will be two ſuch lines as were required.

For a times the ſquare of AB, together with b times the ſquare of AC, is equal to the multiple of the

the ſquare of AE by the ſum of *a*, *b*, together with the multiple of the rectangle BEC by the ſum of *a*, *b*; and *c* times the ſquare of AD, together with the multiple of the ſquare of AE by the ſum of *a*, *b*, is equal to the multiple of the ſquare of AF by the ſum of *a*, *b*, *c*, together with the multiple of the rectangle EFD by the ſum of *a*, *b*, *c*. Therefore *a* times the ſquare of AB, together with *b* times the the ſquare of AC, together with *c* times the ſquare of AD, is equal to the multiple of the ſquare of AF by the ſum of *a*, *b*, *c*, together with the multiple of the rectangle EFD by the ſum of *a*, *b*, *c*, together with the multiple of the rectangle BEC by the ſum of *a*, *b*, that is, equal to the multiple of the ſum of the ſquares of AF, FK (AN), by the ſum of *a*, *b*, *c*, together with the multiple of the rectangle BEC by the ſum of *a*, *b*. Therefore, the ſquare of AB, together with the ſpace to which the ſquare of AC has the ſame ratio that *a* has to *b*, together with the ſpace to which the ſquare of AD has the ſame ratio that *a* has to *c*, is equal to the ſpace to which the ſum of the ſquares of AF, AN, has the ſame ratio that *a* has to the ſum of *a*, *b*, *c*, together with the ſpace to which the rectangle BEC has the ſame ratio that *a* has to the ſum of *a*, *b*, that is, equal to the ſpace to which twice the ſum of the ſquares of AF, AN, has the ſame ratio that twice *a* has to the ſum of *a*, *b*, *c*, together with the ſpace to which twice the rectangle BEC has the ſame ratio that *a* has to the ſum of *a*, *b*. But twice the ſum of the ſquares of AF, AN, is equal to the ſum of the ſquares of AL, AP, by conſtruction, and the rectangle BEC is a given ſpace, and *a*, *b*, are given magnitudes. Therefore, the ſquare of AB, together with the ſpace to which the ſquare of AC has the ſame ratio that *a* has to *b*, together with the ſpace to which the ſquare of AD has the ſame ratio that *a* has to *c*, is equal to the ſpace to which the ſum of the ſquares of AL, AP,

has

has the ſame ratio that twice *a* has to the ſum of *a*, *b*, *c*, together with a given ſpace.

Note. The above is eaſily extenſible to any greater number of lines.

Cor. Let there be any number of right lines given by poſition, that are neither all parallel, nor interſecting each other in one point; two right lines may be found that will be given by poſition, ſuch, that if from any point there be drawn right lines in given angles to all the right lines given by poſition, and likewiſe there be drawn perpendiculars to the two lines found, the ſum of the ſquares of the lines drawn in given angles to the right lines given by poſition, will be equal to the ſpace to which the ſum of the ſquares of the perpendiculars drawn to the two lines found, has a given ratio, together with a given ſpace.

ARTICLE LX.

Animadverſions on ſome Remarks in the Preface to Mr. Howard's *Treatiſe on* Spherical Geometry, *by Mr.* John Lowry.

To the Editor of the Mathematical Repoſitory.

SIR,

I Have obſerved a note at page VIII. of Mr. *Howard's* pompous preface to his *Elements of Spherical Geometry*, lately publiſhed, where he aſſerts that the 1ſt Cor. to Prop. XVI. of *Lucubrations in Spherics*, at page 93 of your *Repoſitory*, is *wrong ſolved as well as that propoſition*. Now, ſir, as I believe that propoſition and its corollaries to be perfectly *true*, and Mr. *Howard's* aſſertion to be *falſe*, I muſt beg leave to make a few remarks on the ſubject: hoping that your known zeal for the cauſe of truth will induce you to allow them a place in the next number of the *Repoſitory*.—This I requeſt, as well for the credit of that work, as for the ſake of my own reputation.—I am, Sir, your humble Servant,

JOHN LOWRY.

Birmingham, May 1, 1798.

MR. HOWARD at Theo. V. Book IV. says, "the greatest great circle spherical triangle AIQ (fig. 199, pl. 14.) that can possibly be contained under two given arches, and any other arch joining their extremities will be when the given arches IA, AQ make right angles with each other." This I deny. I say the the greatest triangle will be that which is inscribed in a circle, the unknown side being the diameter.

To demonstrate his proposition Mr. *Howard* proceeds thus:—"Lay off LA=AQ, and draw through I and LQ the equal parallel circles IB, LQ. With AI as radius, and pole A describe the arch IE, through any point E therein, draw AE, also through Q, E, draw QE and produce it to meet IB in B; join B, O and O, E."

He then proves that the triangle AIQ is greater than the triangle AEQ; and this I grant to be true as long as the point E is taken to the right of AI, for the angle included by the given arches cannot be less than a right angle; but if E be taken on the left of AI Mr. *H's* demonstration fails. This will easily appear by continuing the arch IE to E′, and conceiving the two equal and parallel circles EG, A*m*Q to be drawn, the former touching the arch EIE′ at E′ and meeting AI (continued) in G, and the latter passing through the points A, Q: let AE′, QE′, and QG be joined. Then since the circles IE′, E′G touch at E′ the arch AE′ is perpendicular to the two equal and parallel circles E′G and A*m*Q, therefore the point G falls without the arch E′I and AG is greater than AI, and therefore the triangle QAE′, or its equal the triangle AGQ, is greater than the triangle AIQ. It is also evidently greater than any other triangle on the same base AQ, and whose vertex falls in the arch EIE′; therefore when the triangle is a maximum the arch E′A is not perpendicular to the arch AQ, but to the two equal and parallel circles A*m*Q, E′G. This is also evident from Cor. Prob. VIII. Book VI. or Book II. of the application, for CB (see Mr. *Howard's* figures) will be perpendicular to the two equal and parallel circles *a*V and CIA, but not to the great circle CAQ, as is there very absurdly asserted.

It remains yet to prove that the triangle AE′Q is inscribed in a circle.

Draw QP perpendicular to A*m*Q (fig. 200, pl. 14,) and through the points E′, P describe a great circle, then AE′ is = QP, and AQ=E′MP, therefore the figure AE′MPQA is a rectangle, and the angle E′AQ is the angle of a rectangle, and therefore the triangle AE′Q (by Mr. *H's* own principles) is inscribed in a circle, the unknown side thereof, E′G, is the diameter; consequently the XVI. Prop. of *Spherical Lucubrations* in the *Mathematical Repository* is *true*, and so is the 1st Cor. and the curve that under a given perimeter includes the greatest spherical surface is

a

a *circle*, notwithstanding the properties of that remarkable curve have *all along remained in dark obscurity*, 'till they were discovered by the penetrating ingenuity of Mr. *Howard*: though, by the bye, the properties of that curve were investigated in the *Gentleman's Diary* for the year 1796, by Mr. *Skene* of *Aberdeen*, Mr. *White* of *Dumfries*, and even by two of Mr. *Howard*'s pupils!

The reasoning in several other of Mr. *Howard*'s theorems will, on examination, be found equally fallacious and absurd; and by dividing them into their particular cases, numerous errours will be discovered, which are at present concealed in each common mass. But as I have no inclination of entering into a controversy with that gentleman or any other person (it being the invariable rule of my conduct, and the wish of my heart, to live peaceably with all men), I have declined noticing any of them, hoping that he will correct them in the next edition of his book.

With respect to what Mr. *H.* says at page VIII. of his preface, viz. that "several of *his discoveries* have been published in various periodical publications, particularly the *Mathematical Repository*, by some of his friends and pupils, to whose confidence he communicated his secrets, and trusts his readers will soon be satisfied of the original source of discovery," I must observe that had he pointed out what particular discoveries he laid claim to, I should have been better able to give him an answer. However, that his readers will be capable of tracing his discoveries to their proper origin, I make no dout—at least, such of his readers as are acquainted with the subject;—and for the sake of such as are not, I here mention a few of the many sources from whence these discoveries originated.

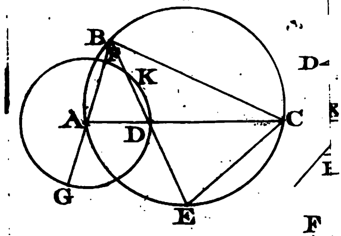
The sources then were—*Euclid's Elements*; *Emerson's Geometry*, and *Trigonometry*; *[illegible]'s* and *Gentleman's Diaries*; *Lawson's Restoration of Apollonius on Tangencies*; a few *Propositions* at the end of *Clarke's Treatise on Circulating Decimals*; Dr. *Hutton's Tracts*; *Simpson's Geometry*, &c. &c. These were the sources! In these the Theorems and Props. are demonstrated and solved in *plano*, and the application of similar principles to spherics was evident, natural and easy.

Compare but Mr. *Howard's* 4th Book and a few Theorems in the 3d No. of the *Repository*, with Mr. *Simpson's Theorems on the Max. and Min.* at the end of his *Geometry*, and I think the origin of discovery will appear evident.

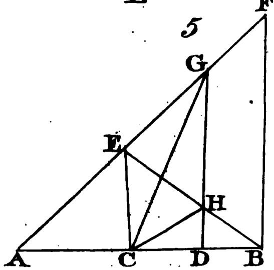
However, notwithstanding what has been said, Mr. *Howard's* book certainly deserves encouragement: the subject is a useful one, and every attempt to simplify and extend it is worthy the patronage of the public.

END OF THE FIRST VOLUME.

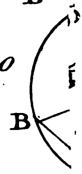
1

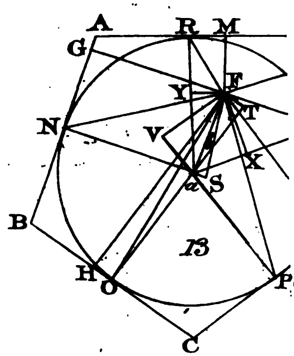


5



10





13

a *circle*, notwithftanding the properties of that remarkable curve have *all along remained in dark obfcurity*, 'till they were difcovered by the penetrating ingenuity of Mr. *Howard*: though, by the bye, the properties of that curve were inveftigated in the *Gentleman's Diary* for the year 1796, by Mr. *Skene* of *Aberdeen*, Mr. *White* of *Dumfries*, and even by two of Mr. *Howard's* pupils!

The reafoning in feveral other of Mr. *Howard's* theorems will, on examination, be found equally fallacious and abfurd; and by dividing them into their particular cafes, numerous errours will be difcovered, which are at prefent concealed in each common mafs. But as I have no inclinatian of entering into a controverfy with that gentleman or any other perfon (it being the invariable rule of my conduct, and the wifh of my heart, to live peaceably with all men). I have declined noticing any of them, hoping that he will correct them in the next edition of his book.

With refpect to what Mr. *H.* fays at page VIII. of his preface, viz. that "feveral of *his difcoveries* have been publifhed in various periodical publications, particularly the *Mathematical Repofitory*, by fome of his friends and pupils, to whofe confidence he communicated his fecrets, and trufts his readers will foon be fatisfied of the original fource of difcovery," I muft obferve that had he pointed out what particular difcoveries he laid claim to, I fhould have been better able to give him an anfwer. However, that his readers will be capable of tracing his discoveries to their proper origin, I make no dout—at leaft, fuch of his readers as are acquainted with the fubject;—and for the fake of fuch as are not, I here mention a few of the many fources from whence thefe difcoveries originated.

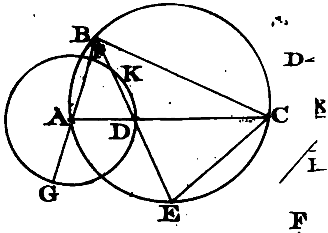
The fources then were—*Euclid's Elements*; *Emerfon's Geometry*, and *Trigonometry*; *Burrow's* and *Gentleman's Diaries*; *Lawson's Reftoration of Apollonius on Tangencies*; a few *Propofitions* at the end of *Clarke's Treatife on Circulating Decimals*; Dr. *Hutton's Tracts*; *Simpfon's Geometry*, &c. &c. Thefe were the fources! In thefe the Theorems and Props. are demonftrated and folved in *plano*, and the application of fimilar principles to fpherics was evident, natural and eafy.

Compare but Mr. *Howard's* 4th Book and a few Theorems in the 3d No. of the *Repofitory*, with Mr. *Simpfon's Theorems on the Max.* and *Min.* at the end of his *Geometry*, and I think the origin of difcovery will appear evident.

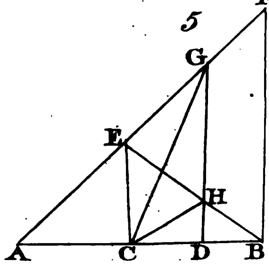
However, notwithftanding what has been faid, Mr. *Howard's* book certainly deferves encouragement: the fubject is a ufeful one, and every attempt to fimplify and extend it is worthy the patronage of the public.

END OF THE FIRST VOLUME.

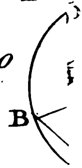
1

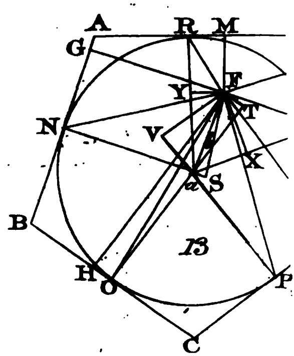


5

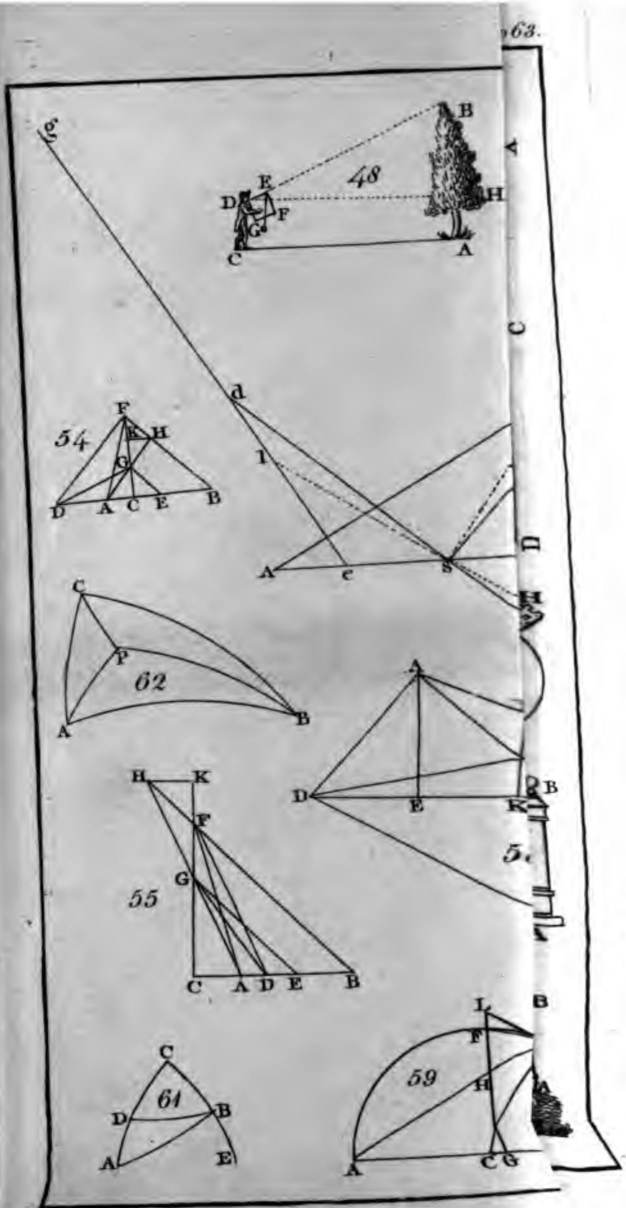


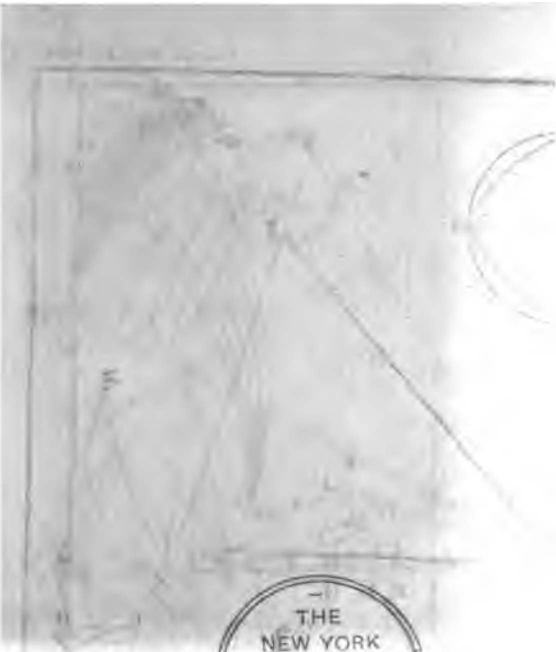
10



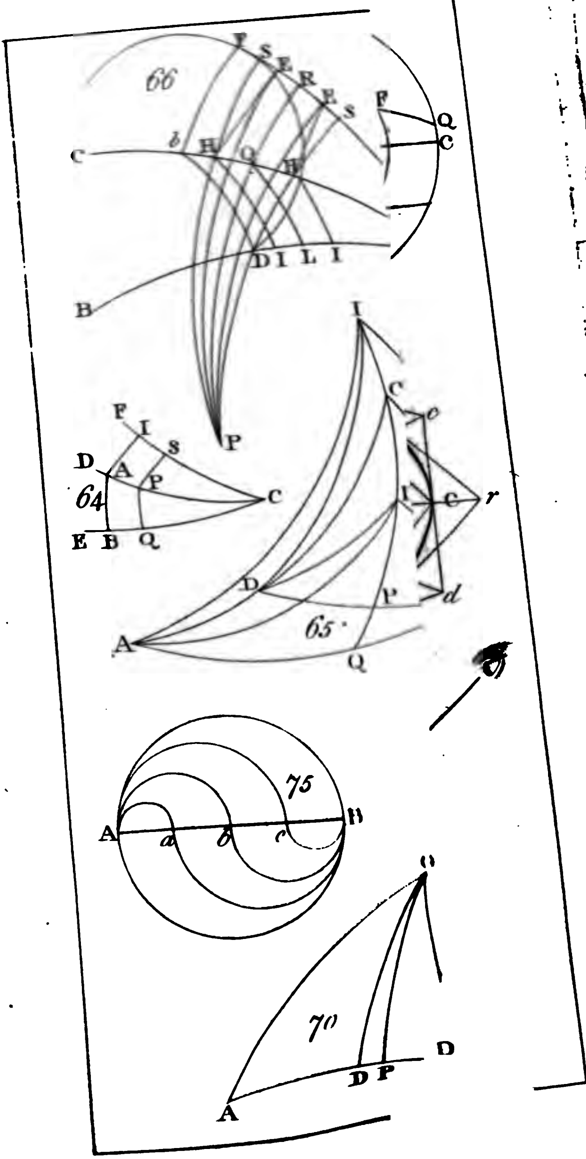


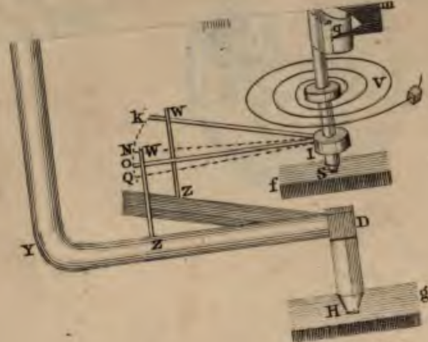
13


63.
48
54
62
55
61
59

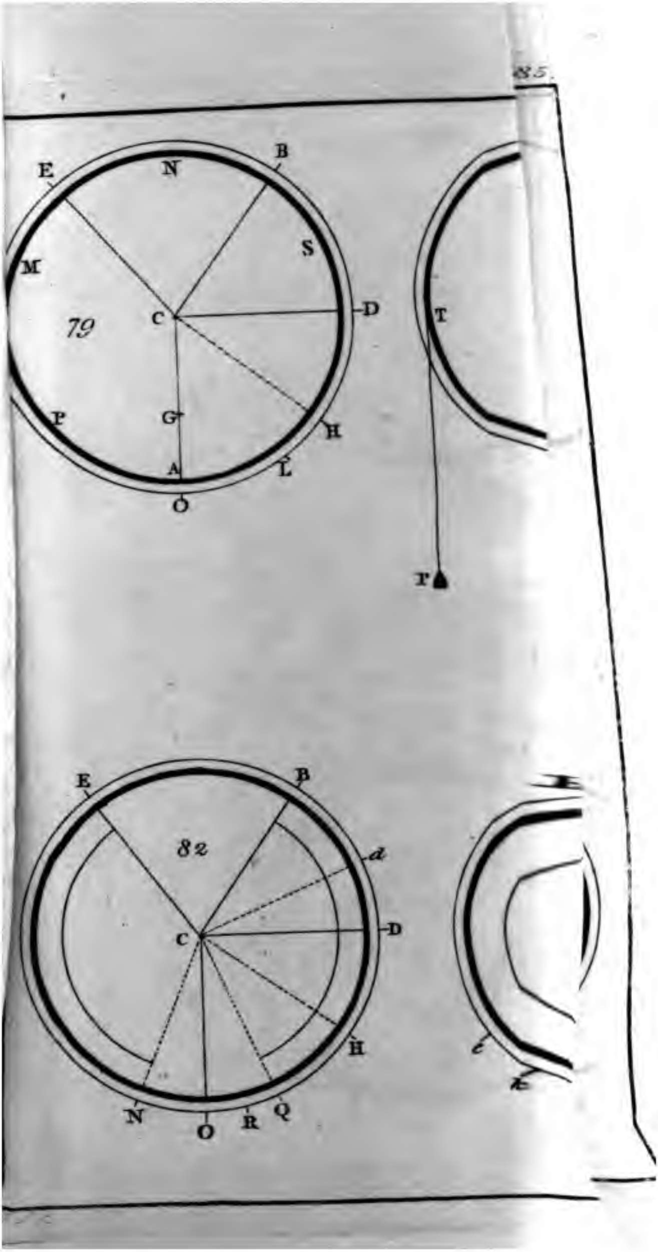


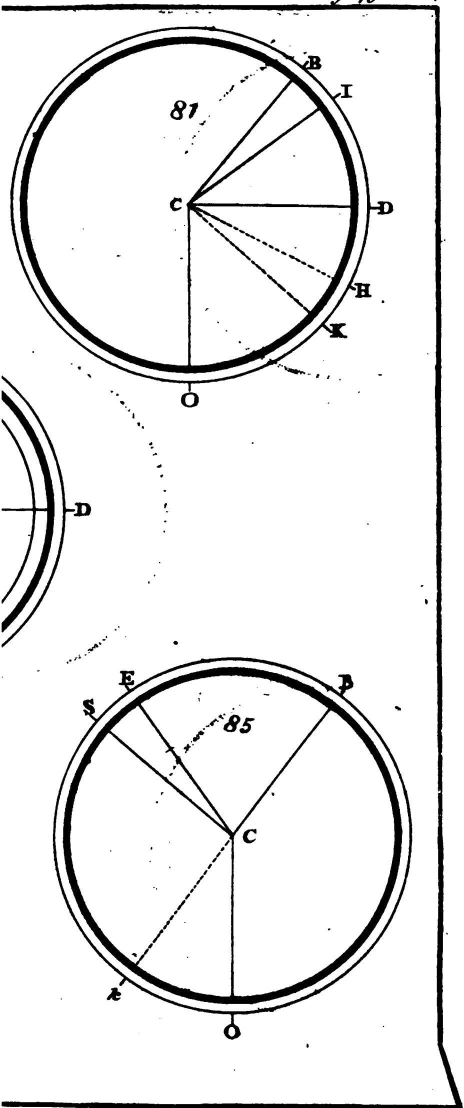
64 to 78

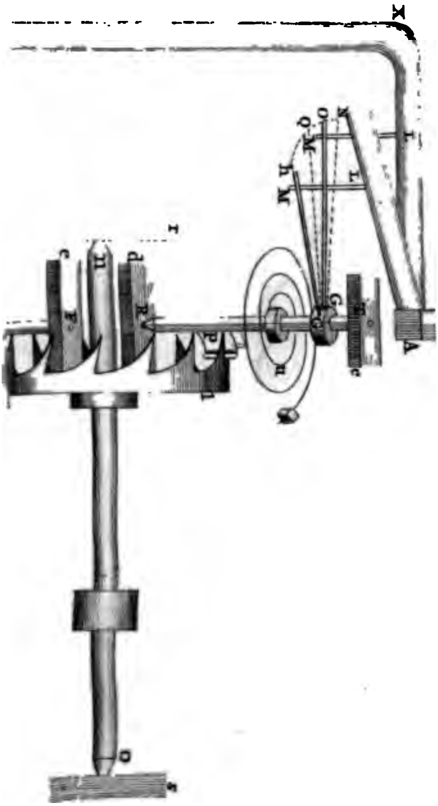

66
F S E R E S
F Q C
C b H Q H
D I L I
B
P
64
F I S
D A P C
E B Q
65
I C c
l C r
D P d
A Q
75
A a b c B
70
D P D
A



J. Swaine, sc.


85
79
E
N
B
S
M
C
D
T
G
P
H
A
L
O
P
82
E
B
d
C
D
H
N
O
R
Q
e
k


81
B
I
C
D
H
K
O
D
E
S
P
85
C
k
O



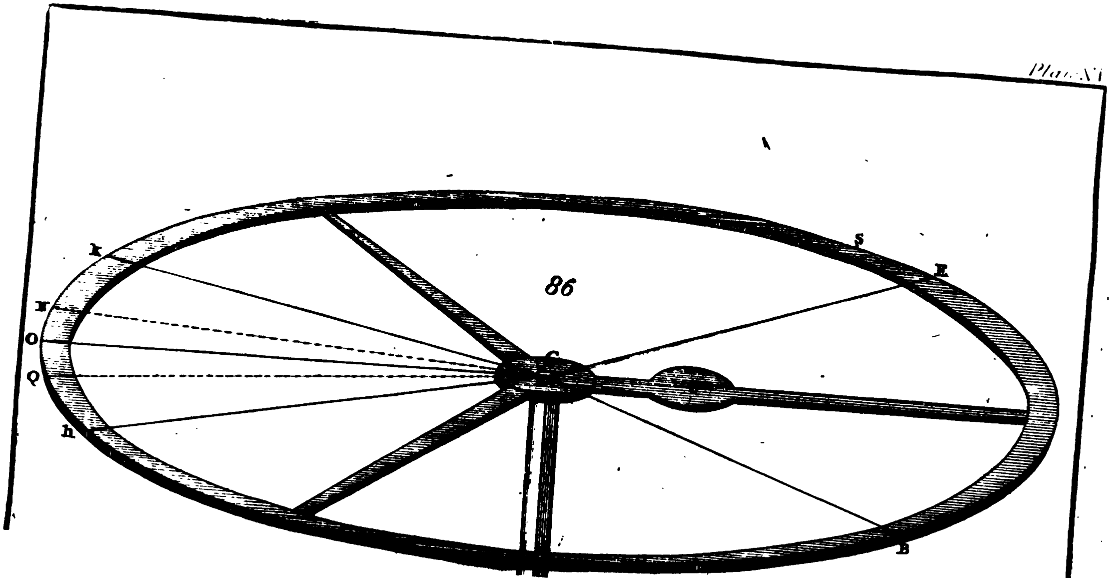
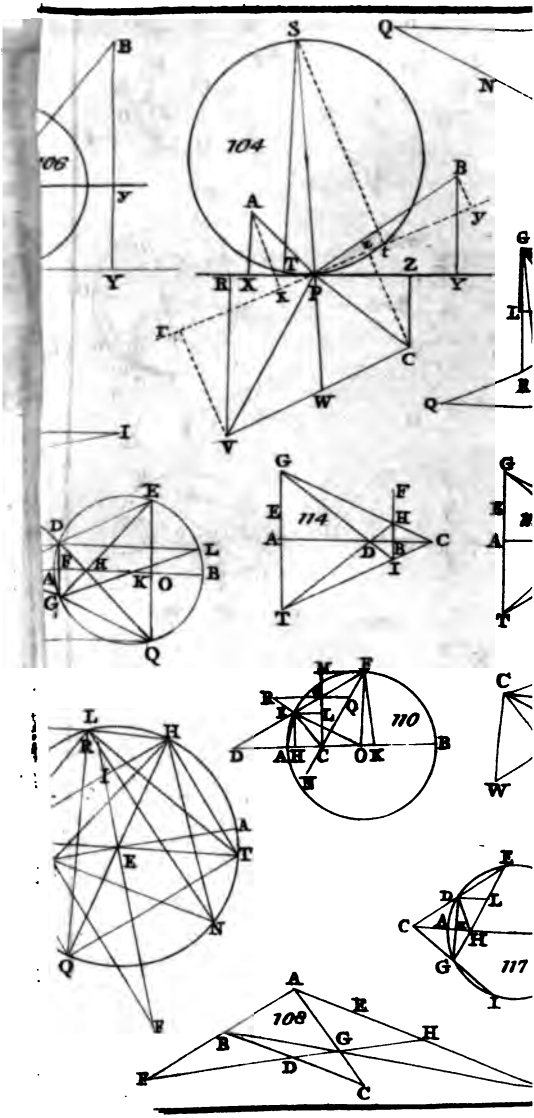
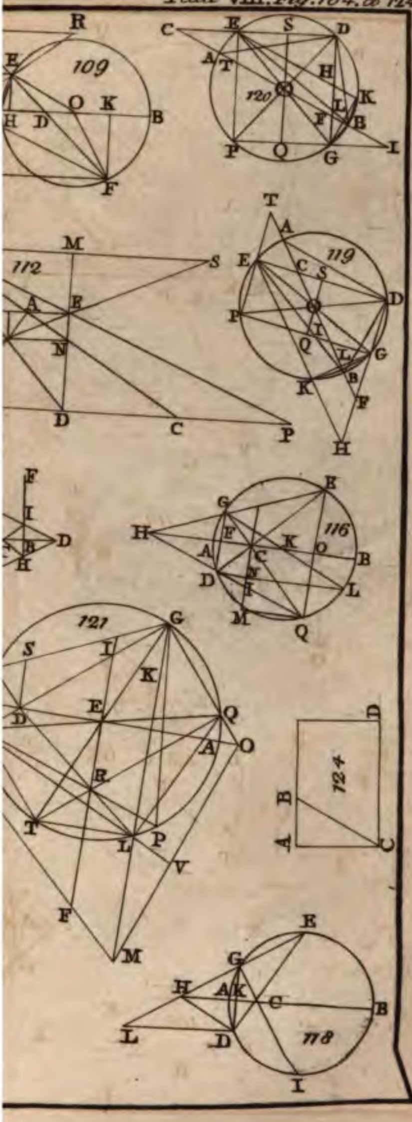


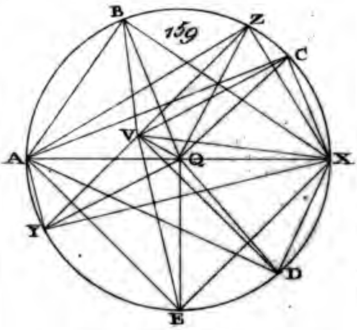
Plate XI

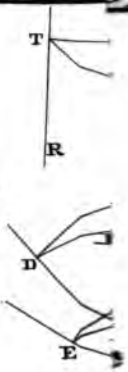


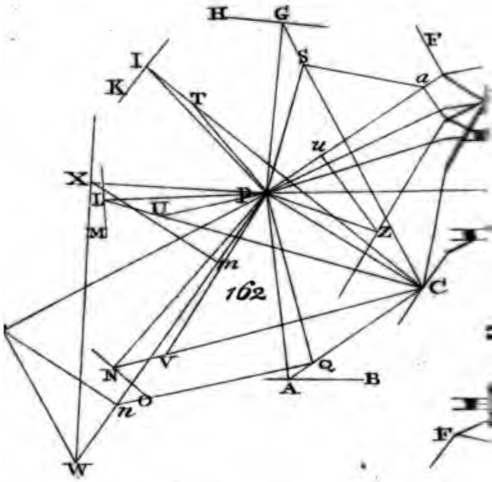
106
104
114
110
117
108

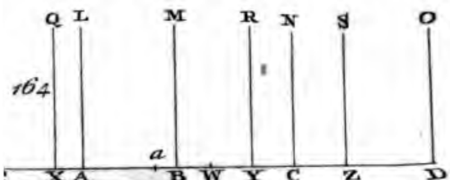


109
R
E
O
K
H
D
B
F
120
C
E
S
D
A
T
H
K
L
F
B
P
O
G
I
112
M
S
A
E
N
D
C
P
119
T
A
E
C
S
D
P
I
Q
L
G
B
K
F
H
F
I
D
B
H
116
G
E
H
F
K
A
C
O
B
D
N
I
L
M
Q
121
G
S
I
K
E
D
Q
A
O
R
T
L
P
V
F
M
124
D
A
B
C
118
E
G
H
A
K
C
B
L
D
I

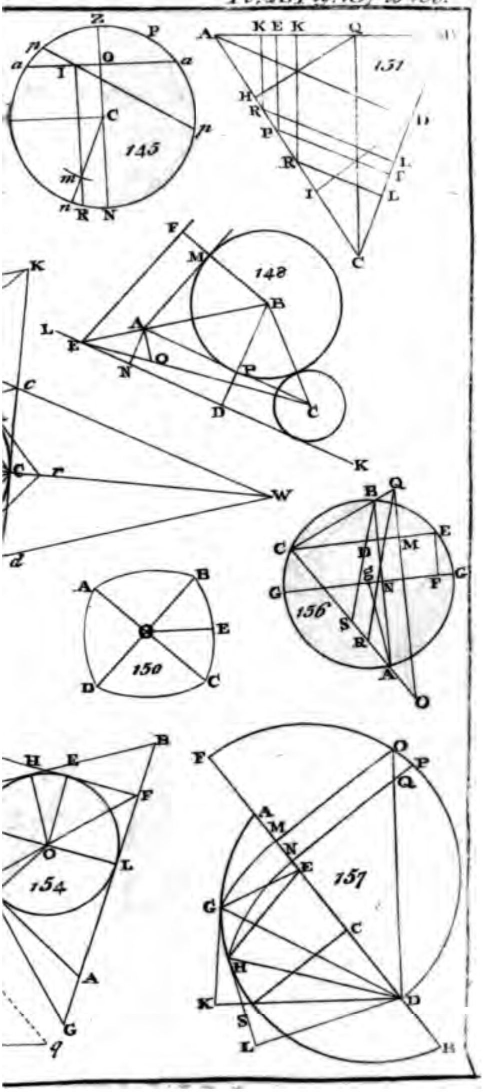

159
B
Z
C
V
A
Q
X
Y
D
E

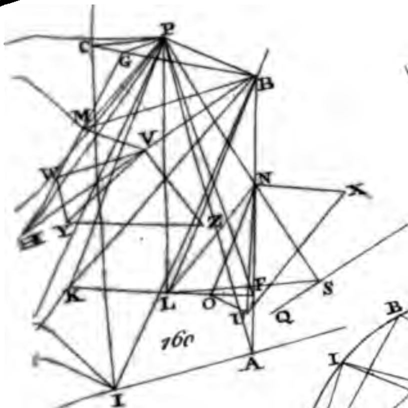

T
R
D
E

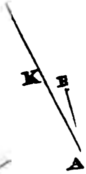

162
H
G
F
S
I
K
T
a
u
X
P
L
U
M
Z
m
C
V
N
Q
A
B
O
n
F
W

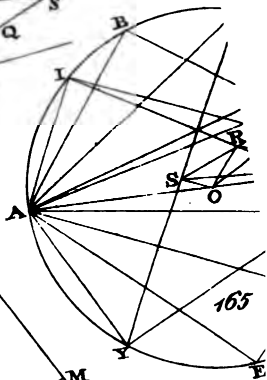

164
Q
L
M
R
N
S
O
a
X
A
B
W
Y
C
Z
D

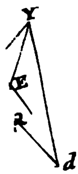
Pl. X. Fig. 137 to 158.

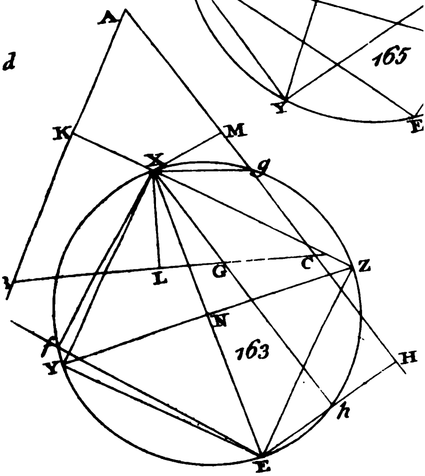


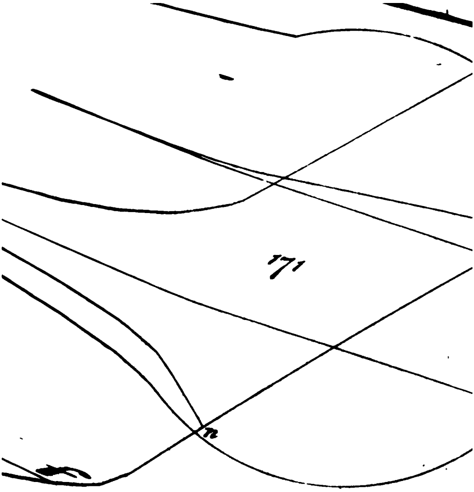

160



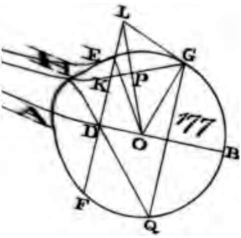

165



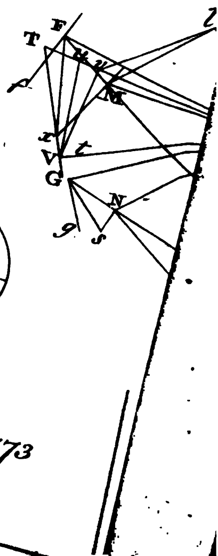

163

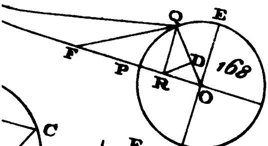


171

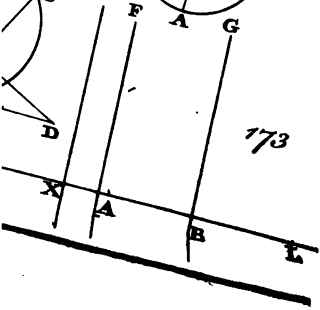


177

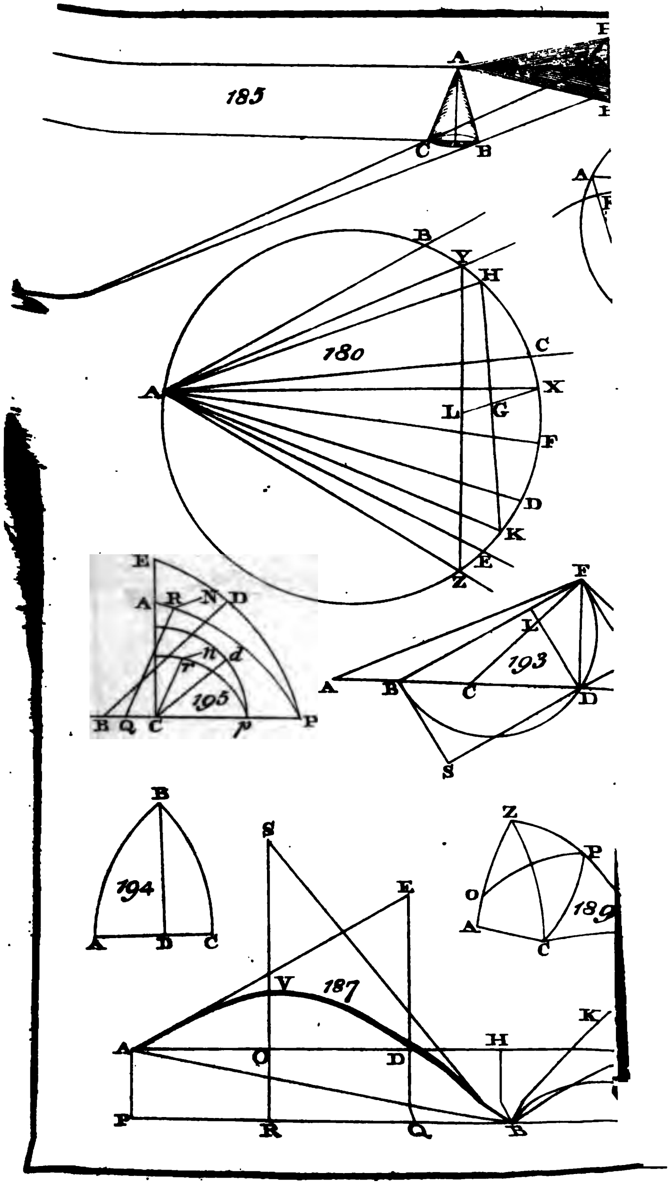


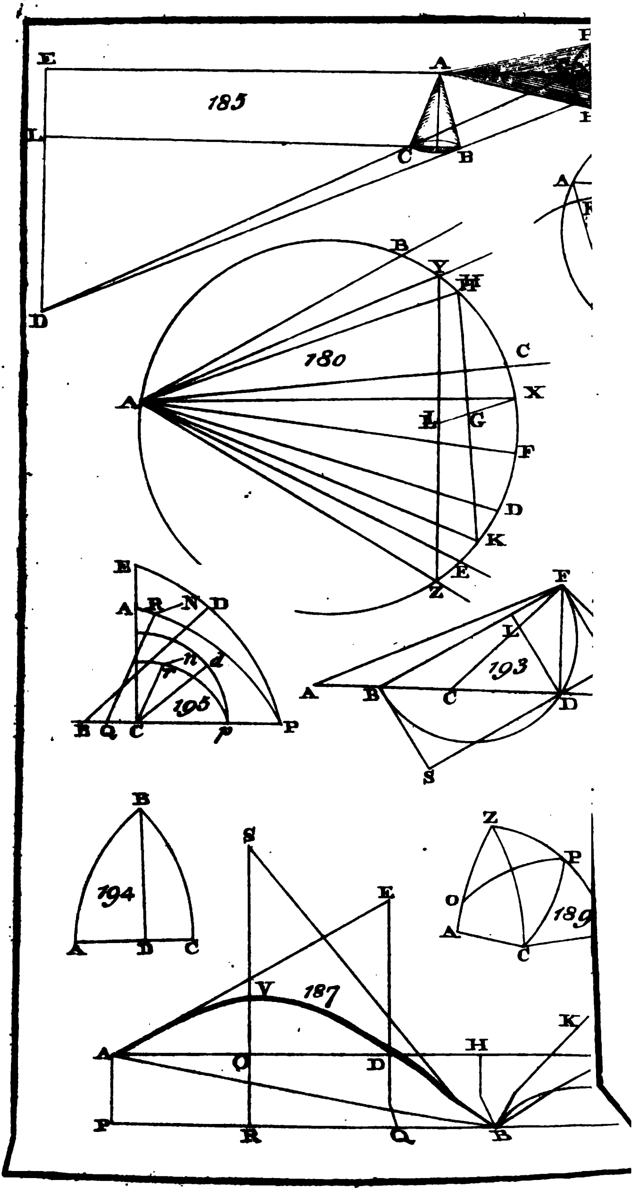


168



173


185
180
195
193
194
187


185
180
195
193
194
189
187

g.5.